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Running-coupling effects in the triple-differential charmless semileptonic decay width

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Abstract: We compute the fully–differential $\bar{B} \to X_u l\bar{\nu}$ decay width to all orders in perturbation theory in the large–β_0 limit. Each of the five structure functions that build the hadronic tensor is expressed as a Borel integral, summing up $\mathcal{O}(C_F\beta_0^{n-1}\alpha_s^n)$ corrections for any n. We derive analytic expressions for the Borel transforms of both real and virtual diagrams with a single dressed gluon, and perform an all–order infrared subtraction, where the Borel parameter serves also as an infrared regulator. Expanding the result we recover the known triple–differential NLO coefficient, and obtain an explicit expression for the $\mathcal{O}(C_F\beta_0\alpha_s^2)$ triple–differential NNLO correction. This result can be used to improve the determination of $|V_{\rm ub}|$ from inclusive $\bar{B} \to X_u l\bar{\nu}$ measurements at the B factories with a variety of kinematic cuts.

Keywords: Weak Decays, Heavy Quark Physics, QCD, Renormalization Regularization and Renormalons.

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1. Introduction

The measurements [1, 2] of semileptonic b decay branching fractions (BF) play a crucial rôle in the determination of the CKM matrix elements [3, 4], which form the basis for many precision tests of the Standard Model and provide an input for new physics searches. While any potential discovery of new physics in the flavor sector is associated with loop–induced transitions, the CKM parameters are most reliably determined by tree–level weak decays. Here two fundamental ingredients are $|V_{\rm cb}|$ and $|V_{\rm ub}|$, which are measured in semileptonic $b \to c$ and $b \to u$ decays, respectively.

Both inclusive and exclusive semileptonic measurements are used to extract these parameters. Inclusive measurements are inherently more robust owing to their limited sensitivity to the hadronic structure of the initial and final states. However, since $b \to u$ transitions are about 50 times less abundant than $b \to c$ ones, kinematic cuts must be applied in order to isolate the $b \to u$ decays and measure $|V_{\rm ub}|$. Consequently, the calculation of the fully differential spectrum is essential for precision measurements of $|V_{\rm ub}|$.

The theoretical calculation of inclusive decay spectra is complicated by the presence of large perturbative and non-perturbative corrections. In b decays into light quarks, e.g. $\bar{B} \to X_s\gamma$ and $\bar{B} \to X_u l\bar{\nu}$, most events are characterized by jet–like momentum configurations, where the invariant mass of the hadronic system in the final state is small. When computing the differential spectrum, or the BF with kinematic cuts, one encounters parametrically–large Sudakov logarithms as well as non-perturbative corrections that are associated with the momentum distribution of the b quark in the meson [5 – 15].

Recently, there has been significant progress in the application of resummed perturbation theory to compute inclusive decay spectra using the method of Dressed Gluon Exponentiation (DGE) [16 – 20]. Underlying this approach is the realization that *running–coupling corrections* play an important rôle in shaping the spectrum. Beyond the purely perturbative issue, infrared renormalons are useful in consistently separating between perturbative and non-perturbative corrections while retaining the predictive power of perturbation theory.

The significance of running–coupling corrections stems from the fact that the gluon virtuality, which sets the *effective scale* of the coupling [21], is typically lower, sometimes significantly lower, than the hard scale m_b that is used as the default renormalization point. Consider for example the fully–integrated $b \to X_u l\bar{\nu}$ width,

$$\Gamma(b \to X_u l\bar{\nu}) = \frac{G_F^2\,|V_{\rm ub}|^2\,m_b^5}{192\pi^3}\left[1 \,+\, b_1\frac{\alpha_s^{(N_f+1)}(m_b)}{\pi} \,+\, b_2\left(\frac{\alpha_s^{(N_f+1)}(m_b)}{\pi}\right)^2 + \cdots\right], \quad (1.1)$$

which is known to NNLO [22],

$$\begin{aligned} b_1 &= C_F\left(\frac{25}{8} - \frac{1}{2}\pi^2\right) \\ b_2 &= \left(\frac{1009}{96} - \frac{77}{72}\pi^2 - 8\zeta_3\right)C_F\beta_0 \\ &+ C_F^2\left(\frac{11047}{2592} + \frac{53}{6}\pi^2\ln(2) - \frac{515}{81}\pi^2 - \frac{223}{36}\zeta_3 + \frac{67}{720}\pi^4\right) \\ &+ C_F\left[\left(\frac{13759}{2592} + \frac{4061}{2592}\pi^2 + \frac{145}{72}\zeta_3 - \frac{53}{12}\pi^2\ln(2) + \frac{101}{1440}\pi^4\right)C_A - \frac{85}{432}\pi^2 + \frac{16987}{1152} - \frac{32}{3}\zeta_3\right], \end{aligned} \quad (1.2)$$

with

$$\beta_0 = \frac{11}{12}C_A - \frac{1}{6}N_f\,, \quad (1.3)$$

where N_f is the number of *light* flavors. Here we have split the b_2 coefficient computed in ref. [22] into a running–coupling part[1], proportional to β_0, and the rest. We find that with $N_f = 4$ the former yields $b_2^{\beta_0} \simeq -26.84$, while the latter $b_2^{\rm rest} \simeq 5.54$, adding up to $b_2 \simeq -21.30$. Evidently, the running–coupling corrections are dominant. These large corrections are related to the leading infrared renormalon of the pole mass m_b, which is located at $u = 1/2$, where u is the relevant Borel parameter. The eventual cancellation [24 – 26] of the corresponding $\mathcal{O}(\Lambda)$ ambiguity in eq. (1.1) involves the overall factor m_b^5 on the one hand and the series in the square brackets on the other. This means, in particular, that higher–order corrections $\mathcal{O}(C_F\beta_0^{n-1}\alpha_s^n)$ in eq. (1.1) are large and form a non-summable series. Owing to the proximity of the $u = 1/2$ renormalon to the origin of the Borel plane, and the relatively low scales involved, the factorial divergence becomes relevant already at the first few orders.

Let us consider now the *differential* decay width. As usual, the effective scale depends on the kinematics; in the region selected by kinematic cuts, where the invariant mass of the hadronic system is small, radiation is confined to be soft or collinear. An obvious consequence is that the effective scale of the coupling gets small, and therefore large running–coupling corrections should be expected.

Moreover, the *normalized* spectrum too is affected by infrared renormalons. Despite the absence of the overall factor m_b^5, infrared renormalons show up in the normalized spectrum because the support of the on-shell decay width is set by m_b: an $\mathcal{O}(\Lambda)$ variation of the pole mass amounts to an $\mathcal{O}(\Lambda)$ *shift* of the Born–level δ–function spectrum. Therefore, *all the moments* of the normalized b decay spectra, defined at the partonic level, have an infrared renormalon ambiguity at $u = 1/2$ [27]. In DGE the corresponding ambiguity cancels between the pole mass and the resummed Sudakov factor upon computing the spectra in physical, hadronic variables [17]; subleading renormalons, $u \geq 3/2$ are present and form the basis for parametrization of power corrections [16 – 18]. In other approaches [28, 29] infrared sensitivity in the "shape function" region is dealt with by using an infrared cutoff in momentum space and absorbing the soft contribution into the definition of the non-perturbative parameters. In any case, the presence of this infrared sensitivity at the level of the partonic calculation cannot be ignored. Computing decay spectra to higher orders in perturbation theory, one therefore expects to find large running–coupling corrections.

Recently, a first complete NNLO calculation of an inclusive decay spectrum has been performed [30, 31] for the case of $\bar{B} \to X_s\gamma$ through the effective magnetic dipole operator. A striking feature of this result is the dominance of the $\mathcal{O}(C_F\beta_0\alpha_s^2)$ contribution (which has been known since a while [32]) with respect to other color factors appearing at this order. The similarity of the two processes, $\bar{B} \to X_s\gamma$ and $\bar{B} \to X_u l\bar{\nu}$, and the dominance of running–coupling corrections in the former, leave no doubt that these corrections are dominant also in the latter.

In the case of the triple–differential $\bar{B} \to X_u l\bar{\nu}$ spectrum, the perturbative expansion is known in full to $\mathcal{O}(\alpha_s)$ (NLO) only [33]. NNLO corrections have been computed in full

[1]The running–coupling part of this coefficient was first computed numerically in ref. [23]. This paper pointed to the low BLM scale characterizing the expansion of the total width in semileptonic decays.

only for the integrated width [22], eq. (1.1) above. In addition, $\mathcal{O}(C_F\beta_0\alpha_s^2)$ real–emission corrections have been recently computed [34] for one particular single–differential spectrum, namely the distribution in the (small) "plus" lightcone–momentum component, p_j^+. The complete $\mathcal{O}(C_F\beta_0^n\alpha_s^{n+1})$ have also been computed numerically for the five structure functions of $\bar{B} \to X_c l\bar{\nu}$ [35]; the results for physical observables obtained with a finite charm mass can in principle be extrapolated numerically to the massless case, but this procedure involves delicate numerical cancellations and lacks the flexibility necessary in practical implementations.

Additional results beyond the NLO are available in the Sudakov limit [16, 17]: the Sudakov exponent has been determined at NNLL accuracy [36 – 38], and to all–orders in the large–β_0 limit [27, 17].

In this paper we perform an all–order calculation of the triple–differential $\bar{B} \to X_u l\bar{\nu}$ spectrum in the large–β_0 limit. We derive analytic expressions for the Borel transform of real and virtual diagrams with a single dressed gluon, which represent the sum of $\mathcal{O}(C_F\beta_0^{n-1}\alpha_s^n)$ corrections for any n. We then preform an all–order infrared subtraction directly in terms of the Borel variable. By expanding the result we recover the known triple–differential NLO coefficient [33], and obtain an explicit expression for the $\mathcal{O}(C_F\beta_0\alpha_s^2)$ triple–differential NNLO correction. By integrating this expression we confirm the results of the single–differential p_j^+ spectrum [34] as well as the β_0 term in the integrated width[22].

The $\mathcal{O}(C_F\beta_0\alpha_s^2)$ triple differential width we compute here is an important ingredient in improving the determination of $|V_{\rm ub}|$ from inclusive $\bar{B} \to X_u l\bar{\nu}$ measurements with a variety of kinematic cuts. In this paper we do not perform any numerical studies; these will be reported on separately.

The structure of the paper is as follows. In section 2 we recall the kinematics and set up the notation. Next, in section 3 we present the real–emission "characteristic function" based on the calculation of ref. [35], which was performed with an off-shell gluon. In section 4 we derive a Borel representation of the real–emission corrections; in section 5 we expand the Borel function to obtain explicit formulae for coefficients at the first few orders. Next, in section 6 we consider the Sudakov limit and extract the non-integrable terms in the real–emission contribution to all orders, confirming the results of [27]. Using the Borel variable as an infrared regulator, we prepare the tools for an all–order infrared subtraction. In section 7 we compute the virtual diagrams using a Borel–modified gluon propagator. We then perform the subtraction of infrared singularities, directly in terms of the Borel variable. In section 8 we combine the results of both real and virtual diagrams for the different structure functions, getting explicit expressions for the coefficients of the triple–differential rate at NNLO. In section 9 we demonstrate the way in which infrared subtraction gets modified depending on the kinematic variables used. Finally, in section 10 we summarize our conclusions.

2. Definitions and kinematics

Let us write the triple–differential width of

$$b(p) \longrightarrow l(p_l) + \bar{\nu}(p_{\bar{\nu}}) + X_u(p_j)$$

as

$$\frac{d\Gamma}{dp_j^+\, dp_j^-\, dE_l} = \frac{G_F^2\,|V_{\rm ub}|^2}{16\pi^3}\, L_{\mu\nu}(p_l, p_{\bar\nu})\mathbb{W}^{\mu\nu}(p,q), \tag{2.1}$$

where the total momentum carried by the leptons is $q = p_l + p_{\bar\nu}$ and E_l is the energy of the charged lepton in the b rest frame. The momentum of the hadronic system is expressed in terms of lightcone components, namely

$$p_j^\pm \equiv E_j \mp |\vec{p}_j|\,, \tag{2.2}$$

so

$$p_j^- = \alpha\, m_b \qquad \text{and} \qquad p_j^+ = \beta\, m_b \tag{2.3}$$

are the large and small lightcone components of the "jet", respectively. They obey

$$0 \le \beta \le \alpha \le 1. \tag{2.4}$$

The relation of these variables with the invariant masses of the "hadronic" (partonic) and the leptonic systems, respectively, is

$$p_j^2 = m_b^2\alpha\beta\,; \qquad\qquad q^2 = m_b^2(1-\alpha)(1-\beta), \tag{2.5}$$

where $0 \le p_j^2 \le \left(1-\sqrt{q^2}\right)^2$. Throughout the paper, m_b represents the pole mass of the b quark.

It is convenient to write the normalization of the differential width in terms of the total tree–level width

$$\Gamma_0 \equiv \frac{G_F^2\,|V_{\rm ub}|^2\; m_b^5}{192\pi^3}, \tag{2.6}$$

namely express eq. (2.1) as:

$$\frac{d^3\Gamma}{d\alpha\, d\beta\, dx} = \frac{1}{2}m_b^3\,\frac{d\Gamma}{dp_j^+\, dp_j^-\, dE_l} = 6\,\Gamma_0\,\frac{1}{m_b^2}\, L_{\mu\nu}(p_l, p_{\bar\nu})\mathbb{W}^{\mu\nu}(p,q), \tag{2.7}$$

where $x \equiv 2E_l/m_b$. Phase–space integration then yields:

$$\begin{aligned}\Gamma_{b\to u} &= \int_0^1 d\alpha \int_0^1 dr\alpha \int_{1-\alpha}^{1-r\alpha} dx\,\frac{d\Gamma}{d\alpha\, d\beta\, dx} \\ &= 6\Gamma_0 \int_0^1 d\alpha \int_0^1 dr\alpha \int_{1-\alpha}^{1-r\alpha} dx\,\frac{1}{m_b^2}\, L_{\mu\nu}(p_l, p_{\bar\nu})\mathbb{W}^{\mu\nu}(p,q),\end{aligned} \tag{2.8}$$

where we use $r \equiv \beta/\alpha$ following [17].

The choice of the lightcone variables in eq. (2.1) is motivated by the fact that the final state is typically jet–like: at Born level $p_j^2 \equiv 0$ so most events are characterized by $p_j^2 \ll m_b^2$, namely $\beta \ll \alpha$. While the first NLO calculation of the triple–differential $b \to u l \bar\nu$ spectrum [33] used other kinematic variables, the advantages of lightcone variables have recently been acknowledged by several authors [28, 34, 14, 17].

For massless leptons, the leptonic tensor is given by:

$$L^{\mu\nu}(p_l, p_{\bar\nu}) = p_l^\mu p_{\bar\nu}^\nu + p_l^\nu p_{\bar\nu}^\mu - p_l\cdot p_{\bar\nu} g^{\mu\nu} - i\epsilon^{\mu\rho\nu\sigma} p_{l\rho} p_{\bar\nu\sigma}. \tag{2.9}$$

Lorentz decomposition of the hadronic tensor $\mathbb{W}^{\mu\nu}(p,q)$ gives rise to five scalar "structure functions":

$$\begin{aligned}\mathbb{W}^{\mu\nu}(p,q) = &-\mathbb{W}_1(\alpha,\beta)g^{\mu\nu} + \mathbb{W}_2(\alpha,\beta)v^\mu v^\nu + i\mathbb{W}_3(\alpha,\beta)\epsilon^{\mu\nu\rho\sigma}v_\rho\hat{q}_\sigma \\ &+ \mathbb{W}_4(\alpha,\beta)\hat{q}_\mu\hat{q}_\nu + \mathbb{W}_5(\alpha,\beta)(v^\mu\hat{q}^\nu + v^\nu\hat{q}^\mu),\end{aligned} \tag{2.10}$$

where $v = p/m_b$ and $\hat{q} = q/m_b$. $\mathbb{W}^{\mu\nu}(p,q)$ is related to $W^{\mu\nu}(p,q)$ defined in [35] by:

$$\mathbb{W}^{\mu\nu}(p,q) = \frac{1}{\pi\, m_b^2}\, W^{\mu\nu}(p,q) \left|\frac{d(q^2, p_j^2)}{d(p_j^+,\, p_j^-)}\right| = \frac{(\alpha-\beta)}{\pi} W^{\mu\nu}(p,q). \tag{2.11}$$

Note that both $W_i^{\mu\nu}(\alpha,\beta)$ and $\mathbb{W}_i^{\mu\nu}(\alpha,\beta)$ are dimensionless. Contracting the Lorentz indices between the leptonic and hadronic tensors, eqs. (2.9) and (2.10) respectively, one finds:

$$\begin{aligned}\frac{1}{m_b^2} L_{\mu\nu}(p_l, p_{\bar{\nu}})\mathbb{W}^{\mu\nu}(p,q) \;=\; &(1-\alpha)(1-\beta)\,\mathbb{W}_1(\alpha,\beta) \\ &- \frac{1}{2}\Big(x^2 \,-\, x\,(2-\alpha-\beta) + (1-\alpha)(1-\beta)\Big)\,\mathbb{W}_2(\alpha,\beta) \\ &+ (1-\alpha)(1-\beta)\left(x - 1 + \frac{1}{2}(\alpha+\beta)\right)\mathbb{W}_3(\alpha,\beta),\end{aligned} \tag{2.12}$$

where, as above, $x = 2E_l/m_b$. Each structure function $\mathbb{W}_i(\alpha,\beta)$ may be decomposed as:

$$\mathbb{W}_i(\alpha,\beta) = V_i(\alpha)\,\delta(\beta) + R_i(\alpha,\beta). \tag{2.13}$$

The functions $V_i(\alpha)$ and $R_i(\alpha,\beta)$ have perturbative expansions in $\alpha_s(m_b)$. At the leading–order (LO), $R_i(\alpha,\beta) = 0$ and, for $i = 1$ to 5,

$$V_i^{\rm LO}(\alpha) = [\alpha,\, 4,\, 2,\, 0,\, -2]\,. \tag{2.14}$$

Substituting (2.13) with (2.14) into (2.12) and using (2.7) one gets:

$$\frac{d^3\Gamma^{\rm LO}}{d\alpha\, d\beta\, dx} = \Gamma_0\,\omega_0(\alpha,x)\,\delta(\beta)\,; \qquad\qquad \omega_0(\alpha,x) \equiv 12\,(x+\alpha-1)\,(2-x-\alpha) \tag{2.15}$$

where Γ_0 is given in (2.6). As usual, this Born–level result receives perturbative corrections from both virtual and real–emission diagrams. Purely virtual contributions are proportional to $\delta(\beta)$. These, however, contain infrared singularities that cancel against soft and collinear real–emission singularities when performing phase–space integration near the $\beta \to 0$ limit. Thus, beyond the LO, the separate definition of $V_i(\alpha)$ and $R_i(\alpha,\beta)$ in (2.13) requires a subtraction prescription. We shall return to this issue in section 6.

The full NLO, $\mathcal{O}(\alpha_s)$ result has been obtained in ref. [33], and checked in various papers that considered higher–order running–coupling effects, including ref. [27] (real emission) and ref. [35] (real and virtual corrections). The result was presented in terms of lightcone variables in ref. [17].

3. Real emission of an off-shell gluon and the characteristic function

Perturbative calculations of many observables in QCD, including inclusive cross sections and decay rates can be improved by the resummation of running–coupling effects [21, 39 – 44, 35]. Specifically, keeping just the leading term in the β function, one sums up the terms proportional to $\beta_0^{n-1}\alpha_s^n$ to all orders, the so called BLM terms.

Technically, running–coupling terms can be conveniently derived using the dispersive method, see e.g. [40, 43, 44], where the one-loop calculation is performed with an off–shell gluon. The calculation of the semileptonic decay "structure functions" $R_i(\alpha,\beta)$ with a single off-shell gluon was performed in ref. [35] for the more general case of $b \to c$ decay, where the charm mass m_c is kept. Here we use this result to derive the corresponding $b \to u$ result by sending $m_c \to 0$. This limit leads to significant simplification that facilitates obtaining closed form analytic expressions.

Further simplification is achieved by choosing the lightcone variables described above, which are suitable for the final state "jet" kinematics. As we shall see, the result for $R_i(\alpha,\beta)$ is completely symmetric under $\alpha \longleftrightarrow \beta$; only the phase–space restriction, eq. (2.4), breaks this symmetry.

The LO calculation of the real–emission diagrams with a gluon of virtuality $m_g^2 = \xi m_b^2$ yields:

$$R_i(\alpha,\beta) \longrightarrow C_F \frac{\alpha_s(m_b)}{\pi} \mathcal{F}_i(\alpha,\beta,\xi) + \mathcal{O}(\alpha_s^2) \tag{3.1}$$

with the following "characteristic function":

$$\mathcal{F}_i(\alpha,\beta,\xi) = \lambda^{-y_i} \left[\frac{1}{(\alpha\beta)^{z_i}} \left(\frac{1}{\tau^-} - \frac{1}{\tau^+} \right) P_i(\alpha,\beta,\xi) \; + \; \frac{1}{\alpha\beta} \ln\left(\frac{\tau^+}{\tau^-} \right) Q_i(\alpha,\beta,\xi) \right] \tag{3.2}$$

where the powers are $y_i = [1,2,1,2,2]$ and $z_i = [3,3,3,2,3]$ for $i = 1$ through 5, respectively, and $P_i(\alpha,\beta,\xi)$ and $Q_i(\alpha,\beta,\xi)$ are polynomials in all their arguments. Finally,

$$\tau^{\pm} = \xi \left(1 - \frac{1}{2\alpha} - \frac{1}{2\beta} \right) - \frac{1}{2}(\alpha+\beta) \pm \frac{1}{2}\sqrt{\lambda} \left(1 - \frac{\xi}{\alpha\beta} \right) \tag{3.3}$$

where

$$\sqrt{\lambda} = \alpha - \beta. \tag{3.4}$$

Note that the phase–space limits are

$$0 \le \xi \le \alpha\beta, \tag{3.5}$$

where the upper limit corresponds to the situation where the entire mass of the hadronic system $p_j^2 = m_b^2 \alpha\beta$ is given by the gluon virtuality.

4. Borel representation

The Borel representation of the result for R_i in the single–dressed–gluon (SDG) approximation is:

$$R_i^{\rm SDG}(\alpha,\beta) = \frac{C_F}{\beta_0} \int_0^\infty du\, T(u) \left(\frac{\Lambda^2}{m_b^2} \right)^u B_i^{\rm SDG}(\alpha,\beta,u) \tag{4.1}$$

$$= C_F \left[c_i^{(1)}(\alpha,\beta)\,\frac{\alpha_s(m_b)}{\pi} + c_i^{(2)}(\alpha,\beta)\,\beta_0 \left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + \cdots \right],$$

where m_b is the bottom pole mass, Λ and α_s are defined in the $\overline{\rm MS}$ scheme, and β_0 is defined in eq. (1.3). $T(u)$ is the inverse Laplace transform of the coupling (see Appendix A); in the large–β_0 limit, where the renormalization–group equation is just one loop, $T(u) \equiv 1$. Resummation of running–coupling corrections beyond this strict limit can also be performed using (4.1). This is briefly explained in Appendix A.

The Borel function in eq. (4.1) can be derived from the following integral over the characteristic function (see e.g. [43, 45] or section 3.1 in [46]),

$$\begin{aligned} B_i^{\rm SDG}(\alpha,\beta,u) &\equiv -{\rm e}^{\frac{5}{3}u}\,\frac{\sin\pi u}{\pi}\int_0^{\alpha\beta}\frac{d\xi}{\xi}\,\mathcal{F}_i(\alpha,\beta,\xi)\,\xi^{-u}, \\ &= -{\rm e}^{\frac{5}{3}u}\,\frac{\sin\pi u}{\pi}\,(\alpha\beta)^{-u}\,b_i(\alpha,\beta,u), \end{aligned} \tag{4.2}$$

where $\mathcal{F}_i(\alpha,\beta,\xi)$ is given in eq. (3.2) and, upon changing variables from ξ to η where $\xi \equiv (1-\eta)\alpha\beta$,

$$b_i(\alpha,\beta,u) \;\equiv\; \int_0^1 d\eta\,(1-\eta)^{-1-u}\,\mathcal{F}_i(\alpha,\beta,(1-\eta)\alpha\beta). \tag{4.3}$$

We have performed the integral in (4.3) analytically, and checked the result by numerical evaluation. Below we give a few details of the calculation and summarize the analytic expressions for $b_i(\alpha,\beta,u)$.

Writing $\tau^\pm$ of eq. (3.3) in terms of η we have $\tau^\pm = (\alpha\beta-\alpha-\beta)(1-\kappa_\pm\eta)$, with

$$\kappa_\pm \equiv \frac{\alpha\beta - \frac{1}{2}(\alpha+\beta\pm\sqrt{\lambda})}{\alpha\beta-\alpha-\beta} \tag{4.4}$$

so using eq. (3.4) $\kappa_+ = (\alpha\beta-\alpha)/(\alpha\beta-\alpha-\beta)$ and $\kappa_- = (\alpha\beta-\beta)/(\alpha\beta-\alpha-\beta)$. Given eq. (3.2), the basic integral needed in eq. (4.3) is of the form

$$\int_0^1 d\eta(1-\eta)^{-u}\frac{1}{1-\kappa_\pm\eta} = \frac{1}{1-u}\;{}_2F_1\Big([1,1],[2-u],\kappa_\pm\Big). \tag{4.5}$$

All the terms in eq. (4.3) can be expressed in terms of this specific hypergeometric function with the two assignments of the argument, $\kappa_\pm$, and some additional rational functions. For example, to integrate the log term in eq. (3.2) times $(1-\eta)^j$ where j is a positive integer (to account for $Q_i(\alpha,\beta,\xi)$ that are quadratic in ξ we need $j=0,1,2$) one first integrates by parts and then uses eq. (4.5) to obtain:

$$\begin{aligned} \int_0^1 d\eta\,(1-\eta)^{-1-u+j}\ln(1-\kappa_\pm\eta) &= \frac{-\kappa_\pm}{j-u}\int_0^1 d\eta\,(1-\eta)^{j-u}\frac{1}{1-\kappa_\pm\eta} \\ &= \frac{-\kappa_\pm}{(j-u)(j-u+1)}\;{}_2F_1\Big([1,1],[2-u+j],\kappa_\pm\Big). \end{aligned} \tag{4.6}$$

Finally, there are known identities that facilitate integer shifts of the indices of hypergeometric functions. For example, to express the hypergeometric function in eq. (4.6) in terms

of our basic one in eq. (4.5) we need to shift the third index from $[2-u+j]$ into $[2-u]$. This is straightforward to do using eq. (2.10) in ref. [47].

The final result for $b_i(\alpha,\beta,u)$ takes the form

$$\begin{aligned} b_i(\alpha,\beta,u) = \lambda^{-y_i}\Bigg[& D_i(\alpha,\beta,u)\Big({}_2F_1\Big([1,1],[2-u],\kappa_+\Big) - {}_2F_1\Big([1,1],[2-u],\kappa_-\Big)\Big) \\ & + \sqrt{\lambda}\, S_i(\alpha,\beta,u)\Big({}_2F_1\Big([1,1],[2-u],\kappa_+\Big) + {}_2F_1\Big([1,1],[2-u],\kappa_-\Big)\Big) \\ & + \sqrt{\lambda}\, T_i(\alpha,\beta,u)\Bigg], \end{aligned} \tag{4.7}$$

where the entire u dependence of the coefficient functions is summarized by

$$\begin{aligned} D_i(\alpha,\beta,u) &= \frac{D_{i,0}(\alpha,\beta)}{u} + \frac{D_{i,1}(\alpha,\beta)}{1-u} + \frac{\widetilde{D}_{i,1}(\alpha,\beta)}{(1-u)^2} + \frac{D_{i,2}(\alpha,\beta)}{2-u} \\ S_i(\alpha,\beta,u) &= \frac{S_{i,0}(\alpha,\beta)}{u} + \frac{S_{i,1}(\alpha,\beta)}{1-u} + \frac{\widetilde{S}_{i,1}(\alpha,\beta)}{(1-u)^2} + \frac{S_{i,2}(\alpha,\beta)}{2-u} \\ T_i(\alpha,\beta,u) &= \frac{T_{i,0}(\alpha,\beta)}{u} + \frac{T_{i,1}(\alpha,\beta)}{1-u} + \frac{T_{i,2}(\alpha,\beta)}{2-u} \end{aligned} \tag{4.8}$$

and where $D_{i,j}(\alpha,\beta)$, $S_{i,j}(\alpha,\beta)$ and $T_{i,j}(\alpha,\beta)$ are rational functions of α and β. The explicit expressions are given in Appendix B. We note that there are simple relations between some of these functions. For example, for any structure function i,

$$D_{i,0}(\alpha,\beta) = (\alpha+\beta-2\alpha\beta)\, S_{i,0}(\alpha,\beta). \tag{4.9}$$

It is straightforward to check that there are no renormalon singularities in $B_i^{\rm SDG}(\alpha,\beta,u)$ of eq. (4.2). As usual, single poles in $b_i(\alpha,\beta,u)$ are cancelled in $B_i^{\rm SDG}(\alpha,\beta,u)$ by the $\sin(\pi u)$ factor, which is associated with the gluon momentum being timelike. The double pole in $b_i(\alpha,\beta,u)$ at $u=1$, which could have resulted in a single pole in $B_i^{\rm SDG}(\alpha,\beta,u)$, is in fact not there: according to eqs. (4.7) and (4.8) its residue is proportional to

$$(2-\kappa_+-\kappa_-)\sqrt{\lambda}\widetilde{S}_{i,1}(\alpha,\beta) + (\kappa_+-\kappa_-)\widetilde{D}_{i,1}(\alpha,\beta) \tag{4.10}$$

and therefore to $(\alpha+\beta)\widetilde{S}_{i,1}(\alpha,\beta) + \widetilde{D}_{i,1}(\alpha,\beta)$, a combination that vanishes for all i.

Thus, $R_i(\alpha,\beta)$ are free of infrared renormalons. Nevertheless, the series for $R_i(\alpha,\beta)$ are divergent and they are Borel summable only for large enough values of β, owing to the convergence constraint on the Borel integral in eq. (4.1) for $u\to\infty$. The consequences have been investigated in detail in the context of the radiative decay [18], see section 2.3 there. In moment space, the convergence constraint is replaced by infrared renormalons [27] through the integration over β near $\beta=0$, see eq. (6.9) below.

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5. Expanding the Borel function

In order to obtain the perturbative coefficients $c_i^{(n)}(\alpha,\beta)$ in the second line of eq. (4.1) one expands the Borel function $B_i^{\rm SDG}(\alpha,\beta,u)$ in powers of u, see eq. (A.3). The expansion of ${}_2F_1\Big([1,1],[2-u],x\Big)$ is known, see e.g. ref. [47]:

$$
\begin{aligned}
{}_2F_1\Big([1,1],[2-u],x\Big) \;=\; \frac{1-u}{x}\Bigg\{ & -\ln(1-x) + u\left[\frac{1}{2}\ln^2(1-x) + \mathrm{Li}_2(x)\right] \\
& + u^2\left[-S_{1,2}(x) - \ln(1-x)\mathrm{Li}_2(x) + \mathrm{Li}_3(x) - \frac{1}{6}\ln^3(1-x)\right] \\
& + u^3\Bigg[- S_{2,2}(x) - \ln(1-x)\mathrm{Li}_3(x) + \ln(1-x)S_{1,2}(x) \\
& \quad + \frac{1}{2}\ln^2(1-x)\mathrm{Li}_2(x) + \frac{1}{24}\ln^4(1-x) + S_{1,3}(x) + \mathrm{Li}_4(x)\Bigg] + \cdots \Bigg\},
\end{aligned}
\tag{5.1}
$$

where Nielsen integrals are defined by

$$
S_{a,b}(x) \equiv \frac{(-1)^{a+b-1}}{(a-1)!b!}\int_0^1 \frac{d\xi}{\xi}\ln^{a-1}(\xi)\ln^b(1-\xi x). \tag{5.2}
$$

Expanding $B_i^{\rm SDG}(\alpha,\beta,u)$ in eq. (4.2) in powers of u, and using eq. (5.1) to expand the hypergeometric functions, we obtain the coefficients, expressed in terms of the functions of Appendix B. At $\mathcal{O}(\alpha_s)$, using the definition of $\kappa_\pm$ in eq. (4.4) and the relation of eq. (4.9), we get:

$$
c_i^{(1)}(\alpha,\beta) = -\lambda^{-y_i}\Bigg\{ -2(\alpha+\beta-\alpha\beta)\ln\left(\frac{\beta}{\alpha}\right)\; S_{i,0}(\alpha,\beta) + \;(\alpha-\beta)\; T_{i,0}(\alpha,\beta),\Bigg\}. \tag{5.3}
$$

At $\mathcal{O}(\beta_0\alpha_s^2)$ we get

$$
\begin{aligned}
c_i^{(2)}(\alpha,\beta) = -\lambda^{-y_i}\Bigg\{ & D_{i,0}(\alpha,\beta)\left[\frac{\frac{1}{2}\ln^2(1-\kappa_+)+\mathrm{Li}_2(\kappa_+)}{\kappa_+} - \frac{\frac{1}{2}\ln^2(1-\kappa_-)+\mathrm{Li}_2(\kappa_-)}{\kappa_-}\right] \\
& + \left(D_{i,1}(\alpha,\beta) - D_{i,0}(\alpha,\beta) + \widetilde{D}_{i,1}(\alpha,\beta) + \frac{1}{2}D_{i,2}(\alpha,\beta)\right)\left(\frac{\ln(1-\kappa_+)}{-\kappa_+} - \frac{\ln(1-\kappa_-)}{-\kappa_-}\right) \\
& + \sqrt{\lambda}\,S_{i,0}(\alpha,\beta)\left[\frac{\frac{1}{2}\ln^2(1-\kappa_+)+\mathrm{Li}_2(\kappa_+)}{\kappa_+} + \frac{\frac{1}{2}\ln^2(1-\kappa_-)+\mathrm{Li}_2(\kappa_-)}{\kappa_-}\right] \\
& + \sqrt{\lambda}\left(S_{i,1}(\alpha,\beta) - S_{i,0}(\alpha,\beta) + \widetilde{S}_{i,1}(\alpha,\beta) + \frac{1}{2}S_{i,2}(\alpha,\beta)\right)\left(\frac{\ln(1-\kappa_+)}{-\kappa_+} + \frac{\ln(1-\kappa_-)}{-\kappa_-}\right) \\
& + \sqrt{\lambda}\left(T_{i,1}(\alpha,\beta) + \frac{1}{2}T_{i,2}(\alpha,\beta)\right)\Bigg\} + \left(\frac{5}{3} - \ln(\alpha\beta)\right)\, c_i^{(1)}(\alpha,\beta).
\end{aligned}
\tag{5.4}
$$

Explicit expressions for $c_i^{(1,2)}(\alpha,\beta)$ that are obtained upon substituting the functions in Appendix B and collecting terms, are listed in Appendix C. In this way it is straightforward to derive higher–order terms in the single–dressed–gluon approximation.

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6. The Sudakov limit

Having derived an explicit result for the Borel function it is straightforward to extract the singular terms in the $\beta \to 0$ limit, the Sudakov limit. The leading terms in this limit have already been computed in ref. [27]. They have been extracted there — see section 3 and appendix B — from an integral representation of the real–emission result for the triple–differential distribution, which was computed directly in the Borel formulation; here we re-derive it from the Borel representation (4.1) of the structure functions computed in the previous section based on the dispersive method.

The Borel function is given in eq. (4.2) where $b_i(\alpha, \beta, u)$ are explicitly written in eq. (4.7). We wish to expand these results at small β keeping the other lightcone variable α as well as the Borel parameter u fixed. In this limit $\kappa_- = \beta(1-\alpha)/\alpha + \mathcal{O}(\beta^2)$ while $\kappa_+ = 1 - \beta/\alpha + \mathcal{O}(\beta^2)$. This means that in eq. (4.3) ${}_2F_1\Big([1,1],[2-u],\kappa_-\Big)$ can be readily expanded at small β while ${}_2F_1\Big([1,1],[2-u],\kappa_+\Big)$ cannot. To extract the leading terms at $\beta \to 0$ from the latter we first use the general identity:

$$ {}_2F_1\Big([1,1],[2-u],\kappa_+\Big) = (1-u)\left[-\frac{1}{u}\,{}_2F_1\Big([1,1],[1+u],1-\kappa_+\Big) + \frac{\pi}{\sin\pi u}(1-\kappa_+)^{-u}\kappa_+^{u-1}\right] \tag{6.1} $$

The new hypergeometric function in eq. (6.1), ${}_2F_1\Big([1,1],[1+u],1-\kappa_+)$, is of course expandable at $\beta \to 0$, while the non-analytic contributions are explicitly given by the second term in eq. (6.1).

With this replacement and using eqs. (4.2), (4.7) and (4.8) and the explicit expressions in Appendix B, we obtain the expected singularity structure [27] (see eq. (3.17) in ref. [17]) for small lightcone component β:

$$ \begin{aligned} B_i^{\rm SDG}(\alpha,\beta,u)|_{\beta\to 0} = {} & \frac{(\alpha\beta)^{-u}\,V_i^{\rm LO}(\alpha)}{\beta}\,\frac{\mathrm{e}^{\frac{5}{3}u}}{u}\times \\ & \left[(1-u)\left(\frac{\beta}{\alpha}\right)^{-u} - \frac{1}{2}\frac{\sin\pi u}{\pi u}\left(\frac{1}{1-u}+\frac{1}{1-u/2}\right)\right]\times\Big(1+\mathcal{O}(\beta/\alpha)\Big), \end{aligned} \tag{6.2} $$

where the last factor in eq. (6.2) serves as a reminder that integrable terms that are suppressed by powers of β are excluded here.

As in the full $R_i(\alpha,\beta)$, there are additional $\mathcal{O}(1/\beta_0)$ contributions to $B_i(\alpha,\beta,u)|_{\beta\to 0}$ starting at $\mathcal{O}(u^1)$, corresponding to $\mathcal{O}(\alpha_s^2)$, the NNLO. These go beyond the large–β_0 limit, and therefore beyond the calculation performed in the present paper. In contrast with the full $B_i(\alpha,\beta,u)$, these $\beta \to 0$ singular terms are *known in full* to the NNLO [37, 16, 17] — see e.g. eq. (3.41) in [17] — and they play an important rôle in Sudakov resummation.

The perturbative expansion in eq. (4.1) corresponding to eq. (6.2) contains log-enhanced terms of the form $c_i^{(n)} \sim \ln^k(\beta)/\beta$, with $k \le n$. Going beyond the large–β_0 limit one finds higher powers of the logarithms owing to multiple gluon emission: at the n-th order one obtains $\ln^k(\beta)/\beta$ where k goes up to $2n-1$, see eq. (6.8) and eq. (6.9) below.

Eq. (6.2) represents the small–β limit of all structure functions except for $i = 4$ where the LO vanishes, so the bremsstrahlung contribution is entirely integrable, $c_4^{(n)} \sim \ln^k(\beta)$,

with $k \leq n$. In the latter case we find the following leading small–β behavior:

$$B_4^{\rm SDG}(\alpha,\beta,u) \simeq \frac{2(\alpha\beta)^{-u}}{\alpha}\frac{{\rm e}^{\frac{5}{3}u}}{u}\left[4(1-u)\left(\frac{\beta}{\alpha}\right)^{-u} - \frac{\sin\pi u}{\pi u}\left(\frac{4}{(1-u)^2} - \frac{2+\alpha}{1-u} + \frac{2+\alpha}{1-u/2}\right)\right]. \tag{6.3}$$

The decay width being infrared and collinear safe, the infrared singularities in eq. (6.2) become integrable near the Sudakov limit once virtual corrections are included. It is therefore convenient to define an integration prescription absorbing singularities from virtual corrections into the real emission part. Defining $r = \beta/\alpha$, we use the plus prescription as follows:

$$\int_0^{r_0} dr F(r)\left[\frac{1}{r^{1+u}}\right]_+ = \frac{F(0)}{u}\Big(1 - r_0^{-u}\Big) + \int_0^{r_0} dr\Big(F(r) - F(0)\Big)\frac{1}{r^{1+u}}, \tag{6.4}$$

where $F(r)$ is a smooth test function. Upon expansion in u, this definition reproduces the $(...)_*$ distribution adopted in [33, 34] and the plus distribution of [35]. Equivalently, we can use:

$$\frac{1}{r^{1+u}} \longrightarrow \left[\frac{1}{r^{1+u}}\right]_+ - \frac{\delta(r)}{u} = \left[\frac{1}{r^{1+u}}\right]_+ - \frac{\alpha\,\delta(\beta)}{u}. \tag{6.5}$$

Having defined the integration prescription, the real–emission coefficients $c_i^{(n)}(\alpha,\beta = r\alpha)$, at each order n in the expansion (4.1), are divided into two parts: the singular part of (6.2) is put under a plus prescription, and the remaining, regular part (which requires no prescription) is left unmodified:

$$c_i^{(n)} \longrightarrow \left[c_i^{(n)\,\rm sing.}\right]_+ + c_i^{(n)\,\rm reg.}. \tag{6.6}$$

Considering in particular the first two orders $c_i^{(1,2)}(\alpha,\beta = r\alpha)$ of eqs. (5.3) and (5.4), which are written explicitly in Appendix C, the part that is put under the plus prescription is:

$$\begin{aligned} c_i^{(1)\,\rm sing.}(\alpha,\beta = r\alpha) &= \frac{V_i^{\rm LO}(\alpha)}{\alpha}\left[-\frac{\ln(r)}{r} - \frac{7}{4}\frac{1}{r}\right] \\ c_i^{(2)\,\rm sing.}(\alpha,\beta = r\alpha) &= \frac{V_i^{\rm LO}(\alpha)}{\alpha}\left[\frac{3}{2}\frac{\ln^2(r)}{r} + \left(2\ln(\alpha) + \frac{13}{12}\right)\frac{\ln(r)}{r}\right. \\ &\quad \left. + \left(\frac{7}{2}\ln(\alpha) + \frac{\pi^2}{6} - \frac{85}{24}\right)\frac{1}{r}\right]. \end{aligned} \tag{6.7}$$

These expressions are consistent with eq. (3.41) in ref. [17], where at the NNLO additional terms, with different color factors, are included.

The subtraction corresponding to this integration prescription will be applied when regularizing the virtual corrections, see eq. (6.12) and section 7 below. It should be emphasized that defining plus distributions with respect to r is just a matter of convention, and depending of the kinematic variables used other choices may turn out more convenient. In section 9 we shortly discuss two alternatives, demonstrating the way in which different infrared–subtraction procedures shuffle terms between the real and virtual contributions.

As discussed in refs. [27, 37, 17], eq. (6.2) contains infrared and collinear singular contributions that are associated with *two* subprocesses, which decouple in the Sudakov limit, the quark distribution (in an on-shell heavy quark) with momentum $\mathcal{O}(m_b\beta)$ and the jet with virtuality $\mathcal{O}(m_b\sqrt{\alpha\beta})$. The two terms in the square brackets in eq. (6.2) correspond to these two Sudakov anomalous dimensions, respectively, which were computed here, again, in the large–β_0 limit.

To perform Sudakov resummation, accounting for *multiple* soft and collinear radiation, we follow [17] and define moments of the structure functions with respect to $r = \beta/\alpha$, in accordance with the plus prescription (6.4):

$$\begin{aligned}\widetilde{\mathbb{W}}_i(\alpha, N) &\equiv \int_0^\alpha d\beta \left(1-\frac{\beta}{\alpha}\right)^{N-1} \mathbb{W}_i(\alpha,\beta) \\ &= H_i(\alpha) \times \mathrm{Sud}(m_b\alpha, N) + \Delta R_i^{(N)}(\alpha),\end{aligned} \tag{6.8}$$

where $H_i(\alpha) = V_i(\alpha)|_{\text{large }\beta_0} + \cdots$. We can deduce the structure of the Sudakov exponent from eq. (6.2):

$$\begin{aligned}\mathrm{Sud}(m_b\alpha, N) = \exp\left\{\frac{C_F}{\beta_0}\int_0^\infty \frac{du}{u}\,T(u)\left(\frac{\Lambda^2}{\alpha^2 m_b^2}\right)^u \right.\Bigg[& B_{\mathcal{S}}(u)\left(\frac{\Gamma(N)\Gamma(-2u)}{\Gamma(N-2u)}+\frac{1}{2u}\right) \\ & \left. -B_{\mathcal{J}}(u)\left(\frac{\Gamma(N)\Gamma(-u)}{\Gamma(N-u)}+\frac{1}{u}\right)\Bigg]\right\},\end{aligned} \tag{6.9}$$

which we wrote as a Borel integral (in the DGE form) with

$$B_{\mathcal{J}}(u)|_{\text{large }\beta_0} = \mathrm{e}^{\frac{5}{3}u}\frac{1}{2}\frac{\sin\pi u}{\pi u}\left(\frac{1}{1-u}+\frac{1}{1-u/2}\right) \tag{6.10}$$

$$B_{\mathcal{S}}(u)|_{\text{large }\beta_0} = \mathrm{e}^{\frac{5}{3}u}(1-u). \tag{6.11}$$

In the exponent in eq. (6.9) we *added*, under the Borel integral

$$\frac{C_F}{\beta_0}\int_0^\infty du\,T(u)\left(\frac{\Lambda^2}{m_b^2}\right)^u \times \Big[\ \dots\ \Big]$$

the singularities that are required for writing the $r\to 0$ non-integrable terms of eq. (6.2) as a plus distribution (eq. (6.4)):

$$\begin{aligned}B[V_0](\alpha,u)|_{\text{sing.}} &\equiv \frac{\alpha^{-2u}}{2u^2}\Big[B_{\mathcal{S}}(u)-2B_{\mathcal{J}}(u)\Big] \\ &= \frac{\alpha^{-2u}\,\mathrm{e}^{\frac{5}{3}u}}{2u^2}\left[(1-u)-\frac{\sin\pi u}{\pi u}\left(\frac{1}{1-u}+\frac{1}{1-u/2}\right)\right] \\ &= -\frac{1}{2u^2}+\left(\ln(\alpha)-\frac{25}{12}\right)\frac{1}{u}-\left(\ln^2(\alpha)-\frac{25}{6}\ln(\alpha)-\frac{1}{6}\pi^2+\frac{245}{72}\right)+\mathcal{O}(u),\end{aligned} \tag{6.12}$$

making the moments in eqs. (6.8) and (6.9) above finite. These terms will be *subtracted* from the virtual corrections in eq. (7.18) below. As shown explicitly in section 7 this subtraction exactly cancels the infrared singularities of the virtual corrections.

7. Virtual corrections

The virtual contribution to the structure functions[2] can be decomposed as in eq. (2.10):

$$V^{\mu\nu}(p,q) = -V_1(\alpha)g^{\mu\nu} + V_2(\alpha)v^\mu v^\nu + iV_3(\alpha)\epsilon^{\mu\nu\rho\sigma}v_\rho\hat{q}_\sigma + V_4(\alpha)\hat{q}_\mu\hat{q}_\nu + V_5(\alpha)(v^\mu\hat{q}^\nu + v^\nu\hat{q}^\mu). \tag{7.1}$$

For each structure function one has an expansion in the coupling, see eq. (7.4) below; according to (2.14), at leading order (Born level) we have:

$$V^{\mu\nu}_{\rm LO}(p,q) = -\alpha g^{\mu\nu} + 4v^\mu v^\nu + i2\epsilon^{\mu\nu\rho\sigma}v_\rho\hat{q}_\sigma - 2(v^\mu\hat{q}^\nu + v^\nu\hat{q}^\mu). \tag{7.2}$$

Let us now compute the virtual corrections in the large–β_0 limit. To this end we modify the gluon propagator according to

$$\frac{g_{\mu\nu}}{-k^2} \;\longrightarrow\; \frac{g_{\mu\nu}}{(-k^2)^{1+u}}. \tag{7.3}$$

With this modification[3], the momentum integration should yield directly the Borel function $B[V_i](\alpha,u)$ in

$$V_i^{\rm SDG}(\alpha) \doteq V_i^{\rm LO}(\alpha) + \frac{C_F}{\beta_0}\int_0^\infty du\,T(u)\left(\frac{\Lambda^2}{m_b^2}\right)^u B[V_i](\alpha,u) \tag{7.4}$$

However, in contrast with the real–emission result of eq. (4.1) that is regular for $u\to 0$, the Borel integral of the virtual diagrams is obstructed by a double pole of $B[V_i](\alpha,u)$ at $u\to 0$, which corresponds to the usual double–logarithmic infrared singularity. Therefore, after computing the momentum integral we shall perform infrared subtraction using the singular terms (6.12). This would finally yield a meaningful Borel representation for the virtual contribution, eq. (7.21) below.

We define $z\equiv q^2/m_b^2$, and in the following, since $\beta=0$, we have $z=1-\alpha$. The result of the virtual diagrams, where the gluon propagator is modified according to (7.3), takes the form[4]:

$$\begin{aligned} B[V^{\mu\nu}](p,q) = {} & B[V_0](\alpha,u)\;\times\; V^{\mu\nu}_{\rm LO}(p,q) \\ & - B[V_1](\alpha,u)\,g^{\mu\nu} \\ & + B[V_2](\alpha,u)\,v^\mu v^\nu \\ & + iB[V_3](\alpha,u)\,\epsilon^{\mu\nu\rho\sigma}v_\rho\hat{q}_\sigma \\ & + B[V_4](\alpha,u)\,\hat{q}_\mu\hat{q}_\nu \\ & + B[V_5](\alpha,u)\,(v^\mu\hat{q}^\nu + v^\nu\hat{q}^\mu) \end{aligned} \tag{7.5}$$

where

$$B[V_0](\alpha,u) = {\rm e}^{\frac{5}{3}u}\left[\frac{1}{2}D(u) + \frac{1}{2}C(z,u) + (1-z)\Big(I_x(z,u) + I_y(z,u) - I_1(z,u)\Big)\right];$$

[2] Our Lorentz decomposition is similar to that of Appendix B in ref. [35]; it differs from that of ref. [33].

[3] See [42] or section 3.1 in [46].

[4] This result agrees with eqs. of (B.11) and (B.12) in ref. [35] (with $m_c=0$). There a gluon mass is introduced instead of a Borel parameter.

$$
\begin{aligned}
B[V_1](\alpha,u) &= -\mathrm{e}^{\frac{5}{3}u}(1-z)\Big(K(z,u)-I_x(z,u)\Big);\\
B[V_2](\alpha,u) &= 4\,\mathrm{e}^{\frac{5}{3}u}\,z\,I_{xy}(z,u);\\
B[V_3](\alpha,u) &= -2\,\mathrm{e}^{\frac{5}{3}u}\,\Big(K(z,u)-I_x(z,u)\Big);\\
B[V_4](\alpha,u) &= -4\,\mathrm{e}^{\frac{5}{3}u}\,\Big(I_x(z,u)-I_{xy}(z,u)\Big);\\
B[V_5](\alpha,u) &= -2\,\mathrm{e}^{\frac{5}{3}u}\,\Big[(1+z)I_{xy}(z,u)-I_x(z,u)\Big].
\end{aligned}
\tag{7.6}
$$

In eq. (7.6) the term $D(u)$

$$
\begin{aligned}
D(u) &= \int_0^1 dx(1-x)^{1+u}x^{-2u}\left(2\frac{1+x}{x}-\frac{1}{u}\right) = -3\frac{1-u}{u}\,\frac{\Gamma(2+u)\Gamma(1-2u)}{\Gamma(3-u)}\\
&= -\frac{3}{2u}-\frac{9}{4}-u\left(\frac{9}{8}+\frac{\pi^2}{2}\right)+O(u^2)
\end{aligned}
\tag{7.7}
$$

is the result of the b-quark self–energy diagram[5], while the u-quark self–energy diagram vanishes in the Borel regularization, as the momentum integral has no scale. All the remaining terms in (7.6) arise from the vertex correction diagram. Let us recall that the Borel parameter regularizes both ultraviolet and infrared logarithmic singularities, just as in dimensional regularization. Thus, $u \to 0$ singularities in individual diagrams arise from both the ultraviolet and the infrared and no distinction is made between them. However, in the present context we know in advance that the ultraviolet divergencies cancel out in the sum of all diagrams — the current is conserved — and therefore the remaining $u \to 0$ singularities in the sum of diagrams are immediately identified as infrared ones. We will address these singularities below.

Let us now briefly describe the calculation of the vertex diagram and define the integrals entering eq. (7.6). Upon combining the propagators using Feynman parametrization, where the b–quark propagator is associated with the Feynman parameter x and the u–quark propagator with y (so the gluon with $1-x-y$), one identifies the scale

$$
M^2 \equiv m_b^2\,x\Big(y(1-z)+x\Big),
\tag{7.8}
$$

where $z \equiv q^2/m_b^2$ as above. Performing next the loop–momentum integral in four dimensions one obtains integrals of the following form over the Feynman parameters:

$$
I_{a,b,c}(z,u) \equiv \int_0^1 dx \int_0^{1-x} dy \frac{x^a\,y^b\,(1-x-y)^u}{\Big[x\Big(y(1-z)+x\Big)\Big]^{c+u}},
\tag{7.9}
$$

where a, b and c are non-negative integers. This integral is computed as follows. One first changes variables into from y into $w = 1-x-y$ and then from x into t where $x = (1-w)(1-t)$. The integration over both w and t extends over the interval $[0,1]$. In these variables the integrand factorizes and, assuming that the parameters a, b and c

[5]Our result for the self–energy diagram agrees with that of ref. [25].

are such that both integrals exist (which is always the case for the required integrals) the result is:

$$I_{a,b,c}(z,u) = \frac{\Gamma(a-c-u+1)\Gamma(b+1)\Gamma(1+u)\Gamma(2+a-2c-2u+b)}{\Gamma(2+a-c-u+b)\Gamma(3+a+b-2c-u)} \times {}_2F_1([c+u,b+1],[2+a-c-u+b],z). \tag{7.10}$$

At the next step, known hypergeometric identities (see e.g. [47]) are used to bring the result into one that is convenient to expand at small u. In all cases it is possible to write the result in terms of a single hypergeometric function ${}_2F_1([1,1+u],[2-u],z)$. This function has the following expansion (type E in ref. [47]):

$$\begin{aligned} {}_2F_1([1,1+u],[2-u],z) \simeq \frac{1-u}{z}\Bigg\{ & -\ln(1-z) - u\Big(-\ln^2(1-z) - \mathrm{Li}_2(z)\Big) \\ & + u^2\Big(-2S_{1,2}(z) - 2\ln(1-z)\mathrm{Li}_2(z) + \mathrm{Li}_3(z) - \frac{2}{3}\ln^3(1-z)\Big) \\ & - u^3\Big(2S_{2,2}(z) + 2\ln(1-z)\mathrm{Li}_3(z) - 4\ln(1-z)S_{1,2}(z) \\ & - 2\ln^2(1-z)\mathrm{Li}_2(z) - \frac{1}{3}\ln^4(1-z) - 4S_{1,3}(z) - \mathrm{Li}_4(z)\Big) + \mathcal{O}(u^4)\Bigg\}. \end{aligned} \tag{7.11}$$

Following ref. [35] (see eq. (B.3) there) we separate the numerator of the vertex diagram into $N_1^{\mu\nu}$, which is composed of terms having powers of the loop momentum in the numerator (*cf.* eq. (B.6) in ref. [35]) and other terms, $N_2^{\mu\nu}$ (*cf.* eq. (B.9) in ref. [35]). In eq. (7.6) above, the $N_1^{\mu\nu}$ gives rise to $C(z,u)$ in $B[V_0](\alpha,u)$:

$$C(z,u) = \frac{2}{u}K(z,u), \tag{7.12}$$

where

$$\begin{aligned} K(z,u) &= \int_0^1 dx \int_0^{1-x} dy \frac{m_b^{2u}(1-x-y)^u}{(M^2)^u} \\ &= \frac{\Gamma(1-2u)\Gamma(1+u)}{\Gamma(3-u)}\left[1 - \frac{u(1-z)}{1-u}\,{}_2F_1([1,1+u],[2-u],z)\right] \\ &= \frac{1}{2} + u\left(\frac{3}{4} + \frac{1-z}{2z}\ln(1-z)\right) + O(u^2)\,, \end{aligned} \tag{7.13}$$

while the $N_2^{\mu\nu}$ is the source of all the other terms, where the following additional integrals

show up:

$$
\begin{aligned}
I_1(z,u) &= \int_0^1 dx \int_0^{1-x} dy \frac{m_b^{2u}(1-x-y)^u}{(M^2)^{1+u}} \\
&= \frac{\Gamma(1+u)\Gamma(1-2u)}{2\,u^2\Gamma(1-u)\,(1-z)}\left[1+\frac{2u\,z}{1-u}\,{}_2F_1([1,1+u],[2-u],z)\right] \\
&= \frac{1}{1-z}\Bigg\{\frac{1}{2\,u^2}\,-\,\frac{\ln(1-z)}{u}\,+\,\frac{\pi^2}{6}+\ln^2(1-z)+\mathrm{Li}_2(z)-\frac{u}{3}\Big(\ln(1-z)\pi^2 \\
&\quad -3\zeta_3+6S_{1,2}(z)+6\ln(1-z)\mathrm{Li}_2(z)-3\mathrm{Li}_3(z)+2\ln^3(1-z)\Big)+\mathcal{O}(u^2)\Bigg\},
\end{aligned}
\tag{7.14}
$$

$$
\begin{aligned}
I_x(z,u) &= \int_0^1 dx \int_0^{1-x} dy \frac{m_b^{2u}(1-x-y)^u\,x}{(M^2)^{1+u}} \\
&= \frac{\Gamma(1-2u)\Gamma(1+u)}{(1-u)\Gamma(2-u)}\,{}_2F_1([1,1+u],[2-u],z) \\
&= -\frac{\ln(1-z)}{z}+\frac{1}{z}\Big(\ln^2(1-z)+\mathrm{Li}_2(z)-\ln(1-z)\Big)\,u+\mathcal{O}(u^2),
\end{aligned}
\tag{7.15}
$$

$$
\begin{aligned}
I_y(z,u) &= \int_0^1 dx \int_0^{1-x} dy \frac{m_b^{2u}(1-x-y)^u\,y}{(M^2)^{1+u}} \\
&= -\frac{\Gamma(1+u)\Gamma(1-2u)}{u\,\Gamma(2-u)(1-z)}\left[1+\frac{u(1+z)}{1-u}\,{}_2F_1([1,1+u],[2-u],z)\right] \\
&= \frac{1}{1-z}\Bigg\{-\frac{1}{u}-1+\left(1+\frac{1}{z}\right)\ln(1-z) \\
&\quad +\left[\left(1+\frac{1}{z}\right)\left(-\ln^2(1-z)+\ln(1-z)-\mathrm{Li}_2(z)\right)-\frac{1}{3}\pi^2-1\right]\,u\,+\mathcal{O}(u^2)\Bigg\},
\end{aligned}
\tag{7.16}
$$

and

$$
\begin{aligned}
I_{xy}(z,u) &= \int_0^1 dx \int_0^{1-x} dy \frac{m_b^{2u}(1-x-y)^u\,xy}{(M^2)^{1+u}} \\
&= -\frac{\Gamma(1+u)\Gamma(1-2u)}{z\Gamma(3-u)}\left[1-\left(1+z-\frac{z}{1-u}\right)\,{}_2F_1([1,1+u],[2-u],z)\right] \\
&= \frac{1}{2z^2}\Bigg\{-\ln(1-z)-z+\Bigg[\ln^2(1-z)+\left(z-\frac{1}{2}\right)\ln(1-z) \\
&\quad -\frac{3}{2}z+\mathrm{Li}_2(z)\Bigg]\,u+\mathcal{O}(u^2)\Bigg\}.
\end{aligned}
\tag{7.17}
$$

Eq. (7.5) is written such that all the $u \to 0$ infrared singularities are in the first term. This term is proportional to the LO result, as must be the case. As mentioned above, in the absence of such singularities one would interpret the expansion of the virtual corrections, starting at $\mathcal{O}(\alpha_s)$, according to eq. (7.4) above. Obviously, since there is a double pole

at $u = 0$, the u-integral (7.4) is ill–defined. One should first perform subtraction of the singularities.

Using the explicit results for the integrals given above, it is straightforward to check that the $u \to 0$ singularities of $B[V_0](\alpha, u)$ in eq. (7.6) do indeed coincide with eq. (6.12), that was determined in section 6 by defining the plus distribution for the real–emission terms. After subtracting from $B[V_0](\alpha, u)$ in eq. (7.6) the terms in eq. (6.12) one has

$$\begin{aligned} B[V^{\mu\nu}](p,q)|_{\rm reg.} = & \Big[B[V_0](\alpha,u) - \; B[V_0](\alpha,u)|_{\rm sing.} \Big] \times V_{\rm LO}^{\mu\nu}(p,q) \\ & - B[V_1](\alpha,u)\, g^{\mu\nu} \\ & + B[V_2](\alpha,u)\, v^\mu v^\nu \\ & + iB[V_3](\alpha,u)\, \epsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma \\ & + B[V_4](\alpha,u)\, \hat{q}_\mu \hat{q}_\nu \\ & + B[V_5](\alpha,u)\, (v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu). \end{aligned} \tag{7.18}$$

Next, let us split the regularized terms proportional to $V_{\rm LO}^{\mu\nu}$ using eq. (7.2) and absorb them into the five different structure functions; we define:

$$\begin{aligned} B[V^{\mu\nu}](p,q)|_{\rm reg.} = & - B[V_1](\alpha,u)|_{\rm reg.}\; g^{\mu\nu} \\ & + B[V_2](\alpha,u)|_{\rm reg.}\; v^\mu v^\nu \\ & + i\, B[V_3](\alpha,u)|_{\rm reg.}\; \epsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma \\ & + B[V_4](\alpha,u)|_{\rm reg.}\; \hat{q}_\mu \hat{q}_\nu \\ & + B[V_5](\alpha,u)|_{\rm reg.}\; (v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu). \end{aligned} \tag{7.19}$$

Finally, using eq. (7.6) with the explicit results for the integrals we get:

$$\begin{aligned} B[V_1](\alpha,u)|_{\rm reg.} = & \; \alpha\, {\rm e}^{\frac{5}{3}u} \Bigg\{ \frac{(2\alpha+u-2)}{u-1}\, \mathbb{G}(u)\, \mathbb{F}(u,\alpha) + \left(\frac{1}{u} + 1 + \frac{1}{2}u - \frac{3}{2}u^2 \right) \mathbb{G}(u) \\ & - \frac{\alpha^{-2u}}{2u^2} \left[1 - u - \frac{\sin \pi u}{\pi u} \left(\frac{1}{1-u} + \frac{1}{1-u/2} \right) \right] \Bigg\}, \\ B[V_2](\alpha,u)|_{\rm reg.} = & \; \frac{4}{\alpha}\; B[V_1](\alpha,u)|_{\rm reg.} + \frac{4\, {\rm e}^{\frac{5}{3}u}\, u\, (1+u)}{1-u}\, \mathbb{G}(u)\, \mathbb{F}(u,\alpha) \\ B[V_3](\alpha,u)|_{\rm reg.} = & \; \frac{2}{\alpha}\; B[V_1](\alpha,u)|_{\rm reg.} \\ B[V_4](\alpha,u)|_{\rm reg.} = & \; \frac{4\,u}{1-\alpha}\, {\rm e}^{\frac{5}{3}u}\, \mathbb{G}(u) \left[\frac{(2\alpha-u-1)}{u-1}\, \mathbb{F}(u,\alpha) + 1 \right], \\ B[V_5](\alpha,u)|_{\rm reg.} = & -\frac{2}{\alpha}\; B[V_1](\alpha,u)|_{\rm reg.} + \frac{2\,u\, {\rm e}^{\frac{5}{3}u}}{1-\alpha}\, \mathbb{G}(u) \left[\frac{2(u+1) - \alpha\,(u+3)}{u-1}\, \mathbb{F}(u,\alpha) - 1 \right], \end{aligned} \tag{7.20}$$

where $\mathbb{G}(u) \equiv -\Gamma(1-2u)\Gamma(u)/\Gamma(3-u)$ and $\mathbb{F}(u,\alpha) \equiv {}_2F_1([1,\, 1+u],\, [2-u],\, 1-\alpha)$.

The infrared–subtracted virtual terms can be expanded order by order in u to get the perturbative corrections in the large–β_0 limit. Let us write, in analogy with the real–emission result (4.1), a Borel representation of $V_i(\alpha)$ in (2.13):

$$\begin{aligned} V_i^{\rm SDG}(\alpha) &= V_i^{\rm LO}(\alpha) + \frac{C_F}{\beta_0}\int_0^\infty du\, T(u)\left(\frac{\Lambda^2}{m_b^2}\right)^u B[V_i](\alpha,u)|_{\rm reg.} \\ &= V_i^{\rm LO}(\alpha) + C_F\left[v_i^{(1)}(\alpha)\frac{\alpha_s(m_b)}{\pi} + v_i^{(2)}(\alpha)\,\beta_0\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + \cdots\right]. \end{aligned} \tag{7.21}$$

The coefficients $v_i^{(1,2)}(\alpha)$ for the five structure functions, $i = 1$ to 5 are listed in Appendix D.

8. The triple–differential width at NLLO in the large–β_0 limit

In the previous sections we computed separately real and virtual corrections to the five different structure functions in the decomposition of the hadronic tensor. For massless leptons, only three of those enter into the expression for the spectrum (2.7) through (2.12). Let us now combine the result into an expression for the triple–differential width. We present explicit expressions up to NNLO which are valid in the on-shell quark mass scheme. Using the results of the previous sections the generalization to higher orders is straightforward.

Let us write the perturbative expansion of the triple–differential width in the large–β_0 limit as follows:

$$\begin{aligned} \frac{1}{\Gamma_0}\frac{d^3\Gamma}{d\alpha\, d\beta\, dx} &= \omega_0(\alpha,x)\,\delta(\beta) \\ &\quad + C_F\left[\frac{\alpha_s(m_b)}{\pi}K_1(\alpha,\beta,x) + \beta_0\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 K_2(\alpha,\beta,x) + \cdots\right], \end{aligned} \tag{8.1}$$

where $\omega_0(\alpha,x)$ is defined in (2.15) and the NLO and NNLO coefficients $K_n(\alpha,\beta,x)$ for $n = 1$ and 2, respectively, will be detailed below. At each order real and virtual contributions to each structure function add up according to (2.13). The coefficients of the triple differential width can therefore be written as follows:

$$K_n(\alpha,\beta,x) = \omega_n(\alpha,x)\,\delta(\beta) + \Big\{K_n^{\rm sing.}(\alpha,\beta,x)\Big\}_+ + K_n^{\rm reg.}(\alpha,\beta,x), \tag{8.2}$$

where, as usual $\beta = \alpha\, r$ and the plus prescription is as defined with respect to r according to eq. (6.4).

To obtain the virtual coefficients $\omega_n(\alpha,x)$ at each order n one substitutes the regularized virtual coefficients of the structure functions (7.21), which are given explicitly in Appendix D for $n = 1$, 2, into eq. (2.12) and uses the result in (2.7). The virtual coefficient at NLO ($n = 1$) is

$$\omega_1(\alpha,x) = -\,\omega_0(\alpha,x)\left(\mathrm{Li}_2(1-\alpha) + \frac{5}{4} + \frac{\pi^2}{3}\right) + 6\,(\alpha - 1 + x)\,(2\,\alpha - 5 + 2\,x)\ln(\alpha) \tag{8.3}$$

and the NNLO result ($n = 2$) is:

$$\begin{aligned}\omega_2(\alpha,x) \;=\; & 6\,(\alpha-1+x)\left[\mathrm{Li}_2(1-\alpha)+\ln^2(\alpha)-\frac{25}{6}\ln(\alpha)\right]\\ & -\omega_0(\alpha,x)\left[\mathrm{Li}_3(1-\alpha)+2\mathrm{Li}_3(\alpha)+\frac{19}{6}\mathrm{Li}_2(1-\alpha)\right.\\ & \left.+\,(\ln(1-\alpha)-1)\ln^2(\alpha)+\left(\frac{4-7\alpha}{6(1-\alpha)}-\pi^2\right)\ln(\alpha)-\zeta_3+\frac{79\pi^2}{72}+\frac{71}{24}\right].\end{aligned} \tag{8.4}$$

Similarly, by substituting the real–emission results for the structure functions into (2.12), one obtains the corresponding real–emission coefficients for the triple differential width. The singular part, $K_n^{\text{sing.}}(\alpha,\beta,x)$, which enters (8.2) under a plus prescription (6.4), is obtained by

$$\begin{aligned}K_n^{\text{sing.}}(\alpha,\alpha\,r,x) \;=& \,6\,(1-\alpha)(1-\alpha\,r)\,c_1^{(n)\,\text{sing.}}(\alpha,\alpha\,r)\\ & -3\left(x^2\;-\;x\,(2-\alpha-\alpha\,r)+(1-\alpha)(1-\alpha\,r)\right)c_2^{(n)\,\text{sing.}}(\alpha,\alpha\,r)\\ & +\;6\,(1-\alpha)(1-\alpha\,r)\left(x-1+\frac{1}{2}(\alpha+\alpha\,r)\right)\;c_3^{(n)\,\text{sing.}}(\alpha,\alpha\,r)\,.\end{aligned} \tag{8.5}$$

Since at any order n the coefficients $c_i^{(n)\,\text{sing.}}(\alpha,\alpha\,r)$ are proportional to the corresponding LO result $V_i^{\text{LO}}(\alpha)$ for $i = 1$ to 3, one obtains the singular part $K_n^{\text{sing.}}(\alpha,\beta,x)$ as the $r \to 0$ singular terms corresponding to the expansion of (6.2) in powers of u, which depend only on r and α, times the following prefactor,

$$\Omega(\alpha,\,r,\,x) \equiv \omega_0(\alpha,x)+6\Big(2\alpha^2-7\alpha-4x+5+2x\alpha\Big)\,\alpha r-6\alpha^2(1-\alpha)\;r^2, \tag{8.6}$$

that depends also on the lepton energy fraction x. In particular, using eq. (6.7), the NLO result is

$$K_1^{\text{sing.}}(\alpha,\,\alpha\,r,\,x) \;=\frac{\Omega(\alpha,\,r,\,x)}{\alpha}\;\left[-\frac{\ln(r)}{r}-\frac{7}{4}\,\frac{1}{r}\right], \tag{8.7}$$

and the NNLO one is:

$$K_2^{\text{sing.}}(\alpha,\alpha\,r,x)=\frac{\Omega(\alpha,\,r,\,x)}{\alpha}\left[\frac{3}{2}\frac{\ln^2(r)}{r}+\left(2\ln(\alpha)+\frac{13}{12}\right)\frac{\ln(r)}{r}+\left(\frac{7}{2}\ln(\alpha)+\frac{\pi^2}{6}-\frac{85}{24}\right)\frac{1}{r}\right]. \tag{8.8}$$

As expected, the $\mathcal{O}(r^0)$ term in $\Omega(\alpha,\,r,\,x)$ coincides with the Born–level result $\omega_0(\alpha,x)$. However, owing to the contraction with the leptonic tensor in (8.5), $\Omega(\alpha,\,r,\,x)$ also contains some $\mathcal{O}(r^1)$ and $\mathcal{O}(r^2)$ terms that generate integrable $\mathcal{O}(r^0)$ and $\mathcal{O}(r^1)$ terms in $K_n^{\text{sing.}}(\alpha,\alpha\,r,x)$. These terms can be freely taken out of the $\{...\}_+$ brackets in (8.2), as they do not vary by applying the plus prescription (6.4).

Finally, the regular terms $K_n^{\text{reg.}}(\alpha,\beta,x)$ are obtained by

$$\begin{aligned}
K_n^{\text{reg.}}(\alpha,\alpha\,r,x) \;=&\, 6\,(1-\alpha)(1-\alpha\,r)\left[c_1^{(n)}(\alpha,\alpha\,r)-c_1^{(n)\,\text{sing.}}(\alpha,\alpha\,r)\right]\\
&-3\left(x^2\;-\;x\,(2-\alpha-\alpha\,r)+(1-\alpha)(1-\alpha\,r)\right)\left[c_2^{(n)}(\alpha,\alpha\,r)-c_2^{(n)\,\text{sing.}}(\alpha,\alpha\,r)\right]\\
&+\;6\,(1-\alpha)(1-\alpha\,r)\left(x-1+\frac{1}{2}(\alpha+\alpha\,r)\right)\;\left[c_3^{(n)}(\alpha,\alpha\,r)-c_3^{(n)\,\text{sing.}}(\alpha,\alpha\,r)\right],
\end{aligned}\tag{8.9}$$

where, for $n=1$ and 2, the explicit expressions for $c_i^{(n)}(\alpha,\beta)$ and its singular part are given in Appendix C and in eq. (6.7), respectively. Using these expressions we find that the regular term at NLO is given by:

$$\begin{aligned}
K_1^{\text{reg.}}(\alpha,\alpha\,r,x) \;=&\,\frac{6\ln(r)}{\alpha\,(1-r)^4}\,\mathbb{Q}_1(\alpha,\,r,\,x)\\
&-\frac{3(1-\alpha)\,(1-\alpha\,r)}{2\,(1-r)}\,\mathbb{Q}_2(\alpha,\,r,\,x)-\frac{3\,(1-x-\alpha)\,(1-x-\alpha\,r)}{(1-r)^3\,\alpha}\,\mathbb{Q}_3(\alpha,\,r,\,x)\\
&-\frac{3\,(1-\alpha)\,(1-\alpha\,r)\,(2\,x-2+\alpha+\alpha\,r)}{2\,\alpha\,(1-r)}\,(4\,\alpha^2\,r-10\,\alpha\,r+7\,r-10\,\alpha+9),
\end{aligned}\tag{8.10}$$

and at NNLO by

$$\begin{aligned}
K_2^{\text{reg.}}(\alpha,\alpha\,r,x) \;=&\;-2\,K_1^{\text{reg.}}(\alpha,\alpha\,r,x)\,\ln(\alpha)-3\,\frac{(1+r-\alpha\,r)\ln(1+r-\alpha\,r)}{\alpha\,r\,(1-\alpha)\,(1-\alpha\,r)\,(1-r)^3}\,\mathbb{P}_1(\alpha,\,r,\,x)\\
&+\frac{1}{2}\,\frac{\ln(r)\,\mathbb{P}_2(\alpha,\,r,\,x)}{(1-\alpha\,r)\,\alpha\,(1-r)^4}-\frac{9\,\mathbb{Q}_1(\alpha,\,r,\,x)}{\alpha\,(1-r)^4}\ln^2(r)+\left(\frac{6\,\mathbb{Q}_1(\alpha,\,r,\,x)}{\alpha\,(1-r)^4}-\frac{\Omega(\alpha,\,r,\,x)}{\alpha\,r}\right)\times\\
&\times\left[\ln(1+r-\alpha\,r)\ln(r)\;+\text{Li}_2\left(\frac{r\,(1-\alpha)}{1+r-\alpha\,r}\right)-\text{Li}_2\left(\frac{1-\alpha\,r}{1+r-\alpha\,r}\right)\right]\\
&+\frac{1}{4}\,\frac{\mathbb{P}_3(\alpha,\,r,\,x)}{(1-r)}+\frac{1}{2}\,\frac{(1-x-\alpha\,r)\,(1-x-\alpha)}{(1-r)^3\,\alpha}\,\mathbb{P}_4(\alpha,\,r,\,x)\\
&+\frac{1}{4}\,\frac{(2\,x-2+\alpha+\alpha\,r)\,(1-\alpha\,r)\,(1-\alpha)}{(1-r)\,\alpha}\,\mathbb{P}_5(\alpha,\,r,\,x)-\frac{\pi^2}{6}\frac{\Omega(\alpha,\,r,\,x)}{\alpha\,r}\,,
\end{aligned}\tag{8.11}$$

where $\Omega(\alpha,\,r,\,x)$ is given in eq. (8.6) and the polynomials $\mathbb{Q}_j(\alpha,\,r,\,x)$ for $j=1$ to 3 and $\mathbb{P}_j(\alpha,\,r,\,x)$ for $j=1$ to 5 are listed in Appendix E.

9. Changing variables: alternative subtraction procedures

In the previous sections we have presented the results for the triple–differential $b\to u l\bar{\nu}$ width to all orders in the large–β_0 limit. We have chosen to describe the hadronic tensor in terms of the lightcone variables α and β and defined plus distributions with respect to $r=\beta/\alpha$. Let us now shortly describe how the results can be used with other kinematic variables. This is often useful for deriving analytic expressions for partially–integrated spectra, as done for example in ref. [33] at the NLO level.

While the result for the virtual diagrams in a given regularization is unique — in the Borel regularization it is given by eqs. (7.6) and (7.5) — the infrared subtraction that

renders their perturbative expansion finite, namely eqs. (7.18) and (7.19), crucially depends on the corresponding real–emission terms that are put under the plus prescription, eq. (6.2), and the variable (r) with respect to which the plus prescription is defined, see eqs. (6.4) and (6.12).

Let us first demonstrate how to use the results of the previous sections in the same kinematic variables α and β, but with a different infrared–subtraction convention. Consider defining the plus distributions *with respect to* β; instead of eq. (6.4) we now write

$$\int_0^{\beta_0} d\beta F(\beta) \left[\frac{1}{\beta^{1+u}}\right]_+ = \frac{F(0)}{u}\Big(1-\beta_0^{-u}\Big) + \int_0^{\beta_0} d\beta \Big(F(\beta)-F(0)\Big)\frac{1}{\beta^{1+u}}, \tag{9.1}$$

that corresponds to the replacement

$$\frac{1}{\beta^{1+u}} \longrightarrow \left[\frac{1}{\beta^{1+u}}\right]_+ - \frac{\delta(\beta)}{u}. \tag{9.2}$$

Taking the corresponding moments and applying (9.1) we get (*cf.* eq. (6.8)):

$$\begin{aligned}\widetilde{\mathbb{W}}_i^{(\beta)}(\alpha,\nu) &\equiv \int_0^1 d\beta\,(1-\beta)^{\nu-1}\,\mathbb{W}_i(\alpha,\beta)\\ &= H_i^{(\beta)}(\alpha)\times \mathrm{Sud}^{(\beta)}(m_b,\alpha,\nu) + \Delta R_i^{(\beta)}(\alpha),\end{aligned} \tag{9.3}$$

where the superscript (β) is used to distinguish the current definition from our default one, and

$$\begin{aligned}\mathrm{Sud}^{(\beta)}(m_b,\alpha,\nu) = \exp\Bigg\{\frac{C_F}{\beta_0}\int_0^\infty \frac{du}{u}\,T(u)\left(\frac{\Lambda^2}{m_b^2}\right)^u \Bigg[B_{\mathcal{S}}(u)\left(\frac{\Gamma(\nu)\Gamma(-2u)}{\Gamma(\nu-2u)}+\frac{1}{2u}\right)\\ -\alpha^{-u}\,B_{\mathcal{J}}(u)\left(\frac{\Gamma(\nu)\Gamma(-u)}{\Gamma(\nu-u)}+\frac{1}{u}\right)\Bigg]\Bigg\},\end{aligned} \tag{9.4}$$

where the large–β_0 anomalous dimensions $B_{\mathcal{S}}(u)$ and $B_{\mathcal{J}}(u)$ are given in (6.10). Note that the explicit α dependence in (9.4) is different from (6.9) owing to the different meaning of the moment variable ν compared to N. The subtraction term $B[V_0](\alpha,u)|_{\mathrm{sing.}(\beta)}$, replacing (6.12), is therefore:

$$\begin{aligned}B[V_0](\alpha,u)\big|_{\mathrm{sing.}(\beta)} &\equiv \frac{1}{2u^2}\Big[B_{\mathcal{S}}(u)-2\,\alpha^{-u}\,B_{\mathcal{J}}(u)\Big] \qquad (9.5)\\ &= \frac{\mathrm{e}^{\frac{5}{3}u}}{2u^2}\left[(1-u)-\alpha^{-u}\,\frac{\sin\pi u}{\pi u}\left(\frac{1}{1-u}+\frac{1}{1-u/2}\right)\right]\\ &= -\frac{1}{2u^2}+\left(\ln(\alpha)-\frac{25}{12}\right)\frac{1}{u}-\left(\frac{1}{2}\ln^2(\alpha)-\frac{29}{12}\ln(\alpha)-\frac{1}{6}\pi^2+\frac{245}{72}\right)+\mathcal{O}(u).\end{aligned}$$

Finally, using (9.5) in eq. (7.18) we get the corresponding infrared–subtracted version of the virtual terms that replaces (7.19) in this alternative convention. The final results for the virtual terms, equivalent to (7.20) immediately follow.

In a similar way one can consider the subtraction using other kinematic variables. A natural choice, which was used in [35] as well as in the original derivation of the singular terms in [27], is based on the invariant masses of the hadronic and the leptonic systems, eq. (2.5). Let us define:

$$\begin{aligned} 1-\xi &= \frac{p_j^2}{m_b^2} = \alpha\beta\,; \qquad\qquad z = \frac{q^2}{m_b^2} = (1-\alpha)(1-\beta) \\ \mathbb{W}_i^{(\xi)}(z,\xi) &= \mathbb{W}_i(\alpha\,,\beta)\left|\frac{d(\alpha,\,\beta)}{d(z,\xi)}\right| = \frac{1}{\alpha-\beta}\mathbb{W}_i(\alpha\,,\beta) \end{aligned} \tag{9.6}$$

Here, infrared singularities are associated with the small jet mass limit $p_j^2 \to 0$,

$$\begin{aligned} B_i^{\rm SDG}(z,\xi,u)|_{\xi\to 1} = \frac{V_i^{\rm LO}(\alpha=1-z)}{(1-\xi)^{1+u}}\frac{\mathrm{e}^{\frac{5}{3}u}}{u}\times \\ \left[(1-u)\left(\frac{1-\xi}{(1-z)^2}\right)^{-u} - \frac{1}{2}\frac{\sin\pi u}{\pi u}\left(\frac{1}{1-u}+\frac{1}{1-u/2}\right)\right]\times\Big(1+\mathcal{O}(1-\xi)\Big)\,, \end{aligned} \tag{9.7}$$

so plus distributions are defined with respect to ξ (i.e. subtracting $\delta(1-\xi)$ terms) and the corresponding moments are:

$$\begin{aligned} \widetilde{\mathbb{W}}_i^{(\xi)}(z,n) &\equiv \int_0^1 d\xi\,\xi^{n-1}\,\mathbb{W}_i^{(\xi)}(z,\xi) \\ &= H_i^{(\xi)}(z)\times \mathrm{Sud}^{(\xi)}(m_b,z,n) + \Delta R_i^{(\xi)}(z), \end{aligned} \tag{9.8}$$

with

$$\begin{aligned} \mathrm{Sud}^{(\xi)}(m_b,z,n) = \exp\left\{\frac{C_F}{\beta_0}\int_0^\infty\frac{du}{u}\,T(u)\,\left(\frac{\Lambda^2}{m_b^2}\right)^u\left[B_{\mathcal{S}}(u)\,(1-z)^{2u}\left(\frac{\Gamma(n)\Gamma(-2u)}{\Gamma(n-2u)}+\frac{1}{2u}\right)\right.\right. \\ \left.\left. -B_{\mathcal{J}}(u)\left(\frac{\Gamma(n)\Gamma(-u)}{\Gamma(n-u)}+\frac{1}{u}\right)\right]\right\}. \end{aligned} \tag{9.9}$$

Therefore, in these variables the subtraction term takes the form

$$B[V_0](z,u)\big|_{\mathrm{sing.}(\xi)} \equiv \frac{1}{2u^2}\Big[(1-z)^{2u}\,B_{\mathcal{S}}(u)\;-\;2\,B_{\mathcal{J}}(u)\Big] \tag{9.10}$$

$$\begin{aligned} &= \frac{\mathrm{e}^{\frac{5}{3}u}}{2u^2}\left[(1-z)^{2u}\,(1-u)\;-\;\frac{\sin\pi u}{\pi u}\left(\frac{1}{1-u}+\frac{1}{1-u/2}\right)\right] \\ &= -\frac{1}{2u^2}\;+\;\left(\ln(1-z)-\frac{25}{12}\right)\frac{1}{u}-\left(-\ln^2(1-z)-\frac{2}{3}\ln(1-z)-\frac{1}{6}\pi^2+\frac{245}{72}\right)+\mathcal{O}(u). \end{aligned}$$

As before, the corresponding infrared–subtracted virtual terms can be obtained using (9.10) in eqs. (7.18) with $\alpha \to 1-z$.

10. Conclusions

We have computed the perturbative expansion of the triple differential width in $b \to X_u l \bar{\nu}$, to all orders in the large–β_0 limit. This is an important step in determining the differential spectrum beyond the NLO.

Several independent partial calculations have been done in the past that provided useful checks. We find complete agreement with the following:

- The NLO calculation of the five structure functions (or the fully differential width) in ref. [33].
- The NNLO result of ref. [22], eq. (1.1) above, where we could check the β_0 piece upon performing phase–space integration according to (2.8).
- The NNLO single–differential distribution with respect to p_j^+, computed in the large–β_0 limit in ref. [34], which we checked by defining the subtraction procedure with respect to $\beta = p_j^+/m_b$ (see section 9) and then integrating over x and α.
- The singularity structure of the real–emission terms as a function of the Borel variable in [27], and the corresponding Sudakov exponent [17].
- The results of ref. [35], which have been used here for the real–emission diagrams and computed by a different method for the virtual ones.

As explained in the introduction, it is expected that the $\mathcal{O}(\beta_0\alpha_s^2)$ contribution computed here constitutes the bulk of the $\mathcal{O}(\alpha_s^2)$ correction. It therefore has an immediate application in improving the calculation of partial branching fractions used in the determination of $|V_{\rm ub}|$ from inclusive measurements in the B–factories with a variety of kinematic cuts.

Although higher–order $\mathcal{O}(\beta_0^{n-1}\alpha_s^n)$ corrections, $n \geq 3$, may also be significant, we do not expect that direct use of the single–dressed–gluon result by itself, namely (4.1) and (7.21) would yield a viable description of the triple differential spectrum. Owing to the $u \to \infty$ convergence constraint, the Borel integral of the real–emission corrections (4.1) does not exist[6] for small p_j^+, namely in the Sudakov region. Better treatment of this region is achieved using moment space [27, 17], where the convergence constraint is replaced by infrared renormalons. Moreover, in the Sudakov region the effect of *multiple* soft and collinear radiation is very important, and can be taken into account by exponentiation (6.8), as done in ref. [17]. The running–coupling corrections computed here can be used to improve the calculation of partial branching fractions in the DGE approach of ref. [17] by incorporating the residual $\mathcal{O}(\beta_0\alpha_s^2)$ correction that is not part of the Sudakov factor into the matching coefficient.

Higher–order running–coupling corrections are also useful for understanding the interplay between perturbative and non-perturbative corrections, and for estimating the latter.

[6]The implications have been studied in detail in ref. [18] in the context of $b \longrightarrow X_s\gamma$; see section 2.3 there.

Our final results for the hadronic tensor, written as analytic functions in the Borel plane, can be used to determine infrared renormalon ambiguities, which are in turn indicative of the form and potential magnitude of power corrections. In the virtual part of (7.21) with (7.20) one identifies infrared renormalon singularities at integer and half integer values of u. In contrast, the differential real–emission contribution, of (4.1) with (4.2), does not present renormalon singularities; as mentioned above these do show up in moment space owing to the integration over p_j^+ near the singular $p_j^+ \to 0$ limit. At the level of the Sudakov exponent this has already been observed in ref. [27] and been put to use in ref. [17]. The present results facilitate the analysis of power corrections over the entire phase space.

Let us end with a brief comment on the technical tools used in this paper, which made it possible to derive analytic expressions for the Borel transform. We exploited two different techniques for the calculation of Feynman diagrams with a single dressed gluon:

- The real–emission diagrams were computed in ref. [35] using the dispersive approach, where the gluon in the final state is assigned a fixed virtuality, which is then used in a dispersive integral with the time–like discontinuity of the coupling. Here we converted the result into a Borel representation and derived analytic expressions for the Borel function.

- The virtual diagrams were computed here directly in terms of the Borel variable, by modifying the gluon propagator according to eq. (7.3), and then preforming the momentum integral.

Having brought the results of both real and virtual diagrams with a single dressed gluon to a common regularization, we could directly perform *an all–order infrared subtraction*. In this novel approach the Borel variable has a double rôle: on the one hand it serves as an infrared regulator for logarithmic singularities — a double pole at $u = 0$, in full analogy with dimensional regularization — and on the other, it serves as a conjugate to $\ln m_b^2/\Lambda^2$, or the inverse of the coupling constant, allowing for all–order resummation of running–coupling corrections.

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A. The scheme–invariant Borel transform

In this paper, e.g. in eqs. (4.1) and (7.21), we use the scheme–invariant [48] Borel representation where $T(u)$ is defined as the inverse Laplace transform of the coupling:

$$\frac{\beta_0 \alpha_s(\mu)}{\pi} = \int_0^\infty du\, T(u) \left(\frac{\Lambda^2}{\mu^2}\right)^u . \tag{A.1}$$

Here the Borel variable u is the Laplace conjugate to the logarithm $\ln \mu^2/\Lambda^2$, rather than to the inverse of the coupling constant, which is used in the standard Borel transform. In this way it is possible to promote the calculation performed in the large–β_0 limit to include

running–coupling terms that are associated with subleading corrections to the β function, in particular β_1. This was used in various applications, see e.g. ref. [49].

In the strict large–β_0 limit, one resums the terms $\beta_0^{n-1}\alpha_s^n$ to any n. In this case $\mathrm{T}(u) \equiv 1$, and eq. (4.1) (or eq. (7.21) for the virtual terms) reduces to the standard Borel transform, with respect to $A_{\overline{\rm MS}}(\mu) = \beta_0 \alpha_s^{\overline{\rm MS}}(\mu)/\pi$, namely

$$\begin{aligned} R_i^{\text{large }\beta_0}(\alpha,\beta) &= \frac{C_F}{\beta_0}\int_0^\infty du\, \exp\Big\{-u/A_{\overline{\rm MS}}(\mu)\Big\}\left(\frac{\mu^2}{m_b^2}\right)^u B_i^{\rm SDG}(\alpha,\beta,u) \qquad \text{(A.2)}\\ &= C_F\left[c_i^{(1)}(\alpha,\beta)\,\frac{\alpha_s(m_b)}{\pi} + c_i^{(2)}(\alpha,\beta)\,\beta_0\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 + \cdots\right], \end{aligned}$$

with the relation:

$$B_i^{\rm SDG}(\alpha,\beta,u) = \sum_{n=1}^{\infty} c_i^{(n)}(\alpha,\beta)\,\frac{u^n}{n!}. \qquad \text{(A.3)}$$

Upon using the scheme–invariant formulation as in eq. (4.1), it is straightforward to include β_1 effects in the running of the coupling by introducing $T(u)$ that corresponds to the Laplace transform of the two–loop coupling (or the 't Hooft–scheme coupling):

$$\begin{aligned} A(\mu) &= \frac{\beta_0\alpha_s^{\text{'t Hooft}}(\mu)}{\pi} = \int_0^\infty du\, T(u)\left(\frac{\Lambda^2}{\mu^2}\right)^u; \qquad \frac{dA}{d\ln\mu^2} = -A^2(1+\delta A),\\ T(u) &= \frac{(u\delta)^{u\delta}\mathrm{e}^{-u\delta}}{\Gamma(1+u\delta)}; \qquad \ln(\mu^2/\Lambda^2) = \frac{1}{A} - \delta\ln\left(1+\frac{1}{\delta A}\right), \qquad \text{(A.4)} \end{aligned}$$

with $\delta \equiv \beta_1/\beta_0^2$. Upon expanding eq. (4.1), or eq. (7.21), with $T(u)$ of (A.4) one obtains, in addition to the large–β_0 terms, also $\beta_1\beta_0^{n-3}\alpha_s^n$ terms, etc.

B. The functions of the lightcone variables entering the Borel transform

Here we give explicit results for the rational functions of the lightcone variables, $D_{i,j}(\alpha,\beta)$, $S_{i,j}(\alpha,\beta)$ and $T_{i,j}(\alpha,\beta)$ entering the Borel function of eq. (4.8). Recall that the first index here, $i = 1$ to 5, corresponds to the structure function, while the second to the location of the singularity on the positive real axis in the Borel plane, $u = j$ with $j = 0$ to 2. There are a few relations among them, valid for any i:

$$D_{i,0}(\alpha,\beta) = (\alpha+\beta-2\,\alpha\,\beta)\,S_{i,0}(\alpha,\beta) \qquad \text{(B.1)}$$

$$\widetilde{D}_{i,1}(\alpha,\beta) = -(\alpha+\beta)\,\widetilde{S}_{i,1}(\alpha,\beta) \qquad \text{(B.2)}$$

$$D_{i,2}(\alpha,\beta) = -\frac{\alpha^3(1-\beta)+\beta^3(1-\alpha)}{\alpha^2+\beta^2+\alpha\beta(1-\alpha-\beta)}\,S_{i,2}(\alpha,\beta) \qquad \text{(B.3)}$$

The remaining functions are:

$$\begin{aligned} S_{1,2}(\alpha,\beta) =& \frac{(-\alpha-\beta+\alpha^2+\beta^2+3\,\alpha\,\beta+3\,\alpha^2\,\beta^2-3\,\alpha^2\,\beta-3\,\alpha\,\beta^2)}{2\,(-1+\beta)\,(-1+\alpha)\,(-\beta-\alpha+\alpha\,\beta)}\times\\ & (-\alpha^2+\alpha^2\,\beta-\alpha\,\beta+\alpha\,\beta^2-\beta^2) \end{aligned} \qquad \text{(B.4)}$$

$$\widetilde{S}_{1,1}(\alpha,\beta) =(-\alpha^2+2\,\alpha^2\,\beta^3+2\,\alpha^3-\beta^2-3\,\alpha^3\,\beta+2\,\beta^3+2\,\alpha^2\,\beta+2\,\alpha^3\,\beta^2-4\,\alpha^2\,\beta^2 + 2\,\alpha\,\beta-3\,\alpha\,\beta^3+2\,\alpha\,\beta^2)/(2\,(-\beta-\alpha+\alpha\,\beta)) \tag{B.5}$$

$$S_{1,1}(\alpha,\beta) =-\beta\,\alpha(-11\,\alpha^3\,\beta+5\,\alpha^3+6\,\alpha^3\,\beta^2+31\,\alpha^2\,\beta-15\,\alpha^2-22\,\alpha^2\,\beta^2+6\,\alpha^2\,\beta^3 - 34\,\alpha\,\beta+12\,\alpha-11\,\alpha\,\beta^3+31\,\alpha\,\beta^2-15\,\beta^2+5\,\beta^3+12\,\beta)/ (4\,(-1+\beta)\,(-1+\alpha)(-\beta-\alpha+\alpha\,\beta)) \tag{B.6}$$

$$S_{1,0}(\alpha,\beta) =\frac{2\,\beta^4-2\,\alpha\,\beta^4+\alpha^2\,\beta^4-4\,\alpha^2\,\beta^3-4\,\alpha^2\,\beta^2-4\,\alpha^3\,\beta^2+\alpha^4\,\beta^2-2\,\alpha^4\,\beta+2\,\alpha^4}{4\,\alpha\,\beta\,(-\beta-\alpha+\alpha\,\beta)} \tag{B.7}$$

$$D_{1,1}(\alpha,\beta) =(3\,\alpha^4\,\beta^3+13\,\alpha^4\,\beta+3\,\alpha^3\,\beta^4-16\,\alpha^2\,\beta^4+13\,\alpha\,\beta^4-16\,\alpha^4\,\beta^2+2\,\alpha^3\,\beta^5+2\,\alpha^5\,\beta^3 + \alpha\,\beta^5+\alpha^5\,\beta-3\,\alpha^5\,\beta^2-3\,\alpha^2\,\beta^5+4\,\alpha^3+11\,\alpha^3\,\beta^2-8\,\alpha^3\,\beta+11\,\alpha^2\,\beta^3 + 8\,\alpha^2\,\beta^2-4\,\alpha^2\,\beta-8\,\alpha\,\beta^3-4\,\alpha\,\beta^2+4\,\beta^3-10\,\alpha^3\,\beta^3+4\,\alpha^4\,\beta^4-4\,\alpha^4 - 4\,\beta^4)/(4\,(-1+\beta)(-1+\alpha)\,(-\beta-\alpha+\alpha\,\beta)) \tag{B.8}$$

$$T_{1,2}(\alpha,\beta) =-(6\,\alpha^4\,\beta^3-8\,\alpha^4\,\beta^2+\alpha^4\,\beta+\alpha^4+9\,\alpha^3\,\beta^2-2\,\alpha^3\,\beta-\alpha^3+6\,\alpha^3\,\beta^4-8\,\alpha^3\,\beta^3 - 6\,\alpha^2\,\beta^2+\alpha^2\,\beta+9\,\alpha^2\,\beta^3-8\,\alpha^2\,\beta^4+\alpha\,\beta^4-2\,\alpha\,\beta^3+\alpha\,\beta^2-\beta^3+\beta^4)/ (4\,(-1+\beta)(-1+\alpha)\,\alpha\,\beta) \tag{B.9}$$

$$T_{1,1}(\alpha,\beta) =(4\,\alpha^4\,\beta^3-8\,\alpha^4\,\beta^2+5\,\alpha^4\,\beta-\alpha^4-12\,\alpha^3\,\beta^3+13\,\alpha^3\,\beta^2-4\,\alpha^3\,\beta+\alpha^3+4\,\alpha^3\,\beta^4 - 8\,\alpha^2\,\beta^4-6\,\alpha^2\,\beta^2-\alpha^2\,\beta+13\,\alpha^2\,\beta^3+5\,\alpha\,\beta^4-4\,\alpha\,\beta^3-\alpha\,\beta^2+\beta^3-\beta^4)/(2 (-1+\beta)\,(-1+\alpha)\,\alpha\,\beta) \tag{B.10}$$

$$T_{1,0}(\alpha,\beta) =\frac{-7\,\alpha\,\beta(\alpha+\beta)-10\,\alpha^3\,\beta-10\,\alpha\,\beta^3+2\,\alpha^2\,\beta^3+7\,\beta^3-4\,\alpha^2\,\beta^2+2\,\alpha^3\,\beta^2+7\,\alpha^3}{4\,\alpha\,\beta} \tag{B.11}$$

$$S_{2,2}(\alpha,\beta) =-2\,(9\,\alpha^2\,\beta^2-9\,\alpha^2\,\beta+\alpha^2-9\,\alpha\,\beta^2+13\,\alpha\,\beta-3\,\alpha+\beta^2-3\,\beta) (-\alpha^2+\alpha^2\,\beta-\alpha\,\beta+\alpha\,\beta^2-\beta^2)/(-\beta-\alpha+\alpha\,\beta) \tag{B.12}$$

$$\widetilde{S}_{2,1}(\alpha,\beta) =-2(9\,\alpha\,\beta^2-2\,\beta^4+39\,\alpha^2\,\beta^3+6\,\alpha^4\,\beta^3+39\,\alpha^3\,\beta^2-34\,\alpha^2\,\beta^2+9\,\alpha^2\,\beta-2\,\alpha^4 + 3\,\alpha^3-20\,\alpha^3\,\beta-13\,\alpha^4\,\beta^2-20\,\alpha\,\beta^3+9\,\alpha\,\beta^4+9\,\alpha^4\,\beta-28\,\alpha^3\,\beta^3-13\,\alpha^2\,\beta^4 + 6\,\alpha^3\,\beta^4+3\,\beta^3)/(-\beta-\alpha+\alpha\,\beta) \tag{B.13}$$

$$S_{2,1}(\alpha,\beta) =(-35\,\alpha^4\,\beta^2+18\,\alpha^4\,\beta^3+17\,\alpha^4\,\beta+91\,\alpha^3\,\beta^2-62\,\alpha^3\,\beta^3+6\,\alpha^3-52\,\alpha^3\,\beta \\ +18\,\alpha^3\,\beta^4-35\,\alpha^2\,\beta^4-88\,\alpha^2\,\beta^2+30\,\alpha^2\,\beta+91\,\alpha^2\,\beta^3+17\,\alpha\,\beta^4-52\,\alpha\,\beta^3 \\ +30\,\alpha\,\beta^2+6\,\beta^3)/(-\beta-\alpha+\alpha\,\beta) \tag{B.14}$$

$$S_{2,0}(\alpha,\beta) =-(-21\,\alpha^4\,\beta^3-2\,\alpha^4\,\beta-21\,\alpha^3\,\beta^4+18\,\alpha^2\,\beta^4-2\,\alpha\,\beta^4+34\,\alpha^3\,\beta^3+4\,\alpha^4\,\beta^4 \\ -14\,\alpha^3\,\beta^2-14\,\alpha^2\,\beta^3+\alpha^5\,\beta^3+4\,\alpha\,\beta^5+18\,\alpha^4\,\beta^2+\alpha^3\,\beta^5-3\,\alpha^2\,\beta^5-2\,\alpha^5 \\ +4\,\alpha^5\,\beta-3\,\alpha^5\,\beta^2-2\,\beta^5)/(\alpha\,\beta\,(-\beta-\alpha+\alpha\,\beta)) \tag{B.15}$$

$$D_{2,1}(\alpha,\beta) =-(-12\,\alpha\,\beta^3-12\,\alpha^3\,\beta+27\,\alpha^3\,\beta^4+14\,\alpha^5\,\beta^3+14\,\alpha^3\,\beta^5-27\,\alpha^2\,\beta^5-27\,\alpha^5\,\beta^2 \\ -4\,\alpha\,\beta^4-20\,\alpha^2\,\beta^4-20\,\alpha^4\,\beta^2+2\,\beta^4+2\,\alpha^4-4\,\alpha^4\,\beta+27\,\alpha^4\,\beta^3+76\,\alpha^3\,\beta^2 \\ +76\,\alpha^2\,\beta^3+13\,\alpha\,\beta^5+13\,\alpha^5\,\beta-106\,\alpha^3\,\beta^3-4\,\alpha^4\,\beta^4 \\ -28\,\alpha^2\,\beta^2)/(-\beta-\alpha+\alpha\,\beta) \tag{B.16}$$

$$T_{2,2}(\alpha,\beta) =(18\,\alpha^4\,\beta^3-24\,\alpha^4\,\beta^2+5\,\alpha^4\,\beta+\alpha^4+25\,\alpha^3\,\beta^2-14\,\alpha^3\,\beta+18\,\alpha^3\,\beta^4-24\,\alpha^3\,\beta^3 \\ +2\,\alpha^2\,\beta^2+25\,\alpha^2\,\beta^3-24\,\alpha^2\,\beta^4+5\,\alpha\,\beta^4-14\,\alpha\,\beta^3+\beta^4)/(\alpha\,\beta) \tag{B.17}$$

$$T_{2,1}(\alpha,\beta) =-2\,(-\beta+2\,\alpha\,\beta-\alpha)\,(-7\,\alpha^3\,\beta+6\,\alpha^3\,\beta^2+\alpha^3+6\,\alpha^2\,\beta^3-16\,\alpha^2\,\beta^2 \\ +11\,\alpha^2\,\beta-7\,\alpha\,\beta^3+11\,\alpha\,\beta^2+\beta^3)/(\alpha\beta) \tag{B.18}$$

$$T_{2,0}(\alpha,\beta) =-(-7\,\beta^4-20\,\alpha^4\,\beta^2-10\,\alpha^3\,\beta-20\,\alpha^2\,\beta^4-10\,\alpha\,\beta^3-56\,\alpha^3\,\beta^3+21\,\alpha^4\,\beta-7\,\alpha^4 \\ -38\,\alpha^2\,\beta^2+57\,\alpha^3\,\beta^2+21\,\alpha\,\beta^4+57\,\alpha^2\,\beta^3+6\,\alpha^4\,\beta^3+6\,\alpha^3\,\beta^4)/(\alpha\,\beta) \tag{B.19}$$

$$S_{3,2}(\alpha,\beta) =-\frac{-\alpha^2+\alpha^2\,\beta-\alpha\,\beta+\alpha\,\beta^2-\beta^2}{-\beta-\alpha+\alpha\,\beta} \tag{B.20}$$

$$\widetilde{S}_{3,1}(\alpha,\beta) =\frac{-3\,\alpha\,\beta^2+2\,\beta^2+2\,\alpha\,\beta+2\,\alpha^2\,\beta^2-3\,\alpha^2\,\beta-\beta+2\,\alpha^2-\alpha}{-\beta-\alpha+\alpha\,\beta} \tag{B.21}$$

$$S_{3,1}(\alpha,\beta) =-\frac{\beta\,\alpha\,(2\,\alpha\,\beta-3\,\alpha+4-3\,\beta)}{2\,(-\beta-\alpha+\alpha\,\beta)} \tag{B.22}$$

$$S_{3,0}(\alpha,\beta) =\frac{-2\,\alpha\,\beta^3+\alpha^3\,\beta^2+\alpha^2\,\beta^3+2\,\alpha^3+2\,\beta^3+2\,\alpha^2\,\beta-6\,\alpha^2\,\beta^2-2\,\alpha^3\,\beta+2\,\alpha\,\beta^2}{2\,\alpha\,\beta\,(-\beta-\alpha+\alpha\,\beta)} \tag{B.23}$$

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$$D_{3,1}(\alpha,\beta) = -\frac{(\alpha^2\,\beta+\alpha\,\beta^2-8\,\alpha\,\beta+4\,\alpha+4\,\beta)\,(-\beta+2\,\alpha\,\beta-\alpha)}{2\,(-\beta-\alpha+\alpha\,\beta)} \tag{B.24}$$

$$T_{3,2}(\alpha,\beta) = \frac{2\,\alpha^2\,\beta+\alpha^2-2\,\alpha\,\beta+2\,\alpha\,\beta^2+\beta^2}{2\,\alpha\,\beta} \tag{B.25}$$

$$T_{3,1}(\alpha,\beta) = \frac{(-\beta+2\,\alpha\,\beta-\alpha)^2}{\alpha\,\beta} \tag{B.26}$$

$$T_{3,0}(\alpha,\beta) = \frac{7\,\alpha^2+7\,\beta^2+4\,\alpha^2\,\beta^2-10\,\alpha\,\beta^2+2\,\alpha\,\beta-10\,\alpha^2\,\beta}{2\,\alpha\,\beta} \tag{B.27}$$

$$\begin{aligned} S_{4,2}(\alpha,\beta) = &-(-\alpha^2+\alpha^2\,\beta-\alpha\,\beta+\alpha\,\beta^2-\beta^2)(4\,\alpha^2-6\,\alpha-6\,\beta+4\,\beta^2+22\,\alpha\,\beta+12\,\alpha^2\,\beta^2 \\ &-18\,\alpha^2\,\beta-18\,\alpha\,\beta^2+3\,\alpha\,\beta^3+3\,\alpha^3\,\beta)/((-1+\beta)\,(-1+\alpha)\,(-\beta-\alpha+\alpha\,\beta)) \end{aligned} \tag{B.28}$$

$$\begin{aligned} \widetilde{S}_{4,1}(\alpha,\beta) = &-2(2\,\beta^3+5\,\alpha^2\,\beta^3+2\,\alpha^3+10\,\alpha^2\,\beta-8\,\alpha\,\beta^3+5\,\alpha^3\,\beta^2-14\,\alpha^2\,\beta^2+10\,\alpha\,\beta^2 \\ &+\alpha\,\beta^4+\alpha^4\,\beta-8\,\alpha^3\,\beta)/(-\beta-\alpha+\alpha\,\beta) \end{aligned} \tag{B.29}$$

$$\begin{aligned} S_{4,1}(\alpha,\beta) = &(195\,\alpha^4\,\beta^3+78\,\alpha^4\,\beta+195\,\alpha^3\,\beta^4-187\,\alpha^2\,\beta^4+78\,\alpha\,\beta^4-187\,\alpha^4\,\beta^2-51\,\alpha^3\,\beta^5 \\ &-51\,\alpha^5\,\beta^3-28\,\alpha\,\beta^5-28\,\alpha^5\,\beta+60\,\alpha^5\,\beta^2+15\,\alpha^5\,\beta^4+15\,\alpha^4\,\beta^5+60\,\alpha^2\,\beta^5 \\ &+6\,\alpha^3+3\,\alpha^3\,\beta^6+3\,\alpha^6\,\beta^3+3\,\alpha^6\,\beta+3\,\alpha\,\beta^6-6\,\alpha^2\,\beta^6-6\,\alpha^6\,\beta^2+4\,\beta^5+4\,\alpha^5 \\ &+254\,\alpha^3\,\beta^2-84\,\alpha^3\,\beta+254\,\alpha^2\,\beta^3-148\,\alpha^2\,\beta^2+30\,\alpha^2\,\beta-84\,\alpha\,\beta^3+30\,\alpha\,\beta^2 \\ &+6\,\beta^3-326\,\alpha^3\,\beta^3-90\,\alpha^4\,\beta^4-10\,\alpha^4-10\,\beta^4)\Big/((-\beta-\alpha+\alpha\,\beta)\,(-1+\alpha)^2 \\ &(-1+\beta)^2) \end{aligned} \tag{B.30}$$

$$S_{4,0}(\alpha,\beta) = -\frac{2\,(-2\,\alpha^3+\alpha\,\beta^3-2\,\beta^3-7\,\alpha^2\,\beta+\alpha^3\,\beta+\alpha^2\,\beta^2-7\,\alpha\,\beta^2)}{-\beta-\alpha+\alpha\,\beta} \tag{B.31}$$

$$\begin{aligned} D_{4,1}(\alpha,\beta) = &-(3\,\alpha^7\,\beta+207\,\alpha^4\,\beta^3+6\,\alpha^4\,\beta+207\,\alpha^3\,\beta^4-116\,\alpha^2\,\beta^4+6\,\alpha\,\beta^4-116\,\alpha^4\,\beta^2 \\ &-29\,\alpha^3\,\beta^5-29\,\alpha^5\,\beta^3+22\,\alpha\,\beta^5+22\,\alpha^5\,\beta+5\,\alpha^5\,\beta^2+5\,\alpha^2\,\beta^5+5\,\alpha^5\,\beta^4+5\,\alpha^4\,\beta^5 \\ &+5\,\alpha^2\,\beta^5+3\,\alpha^3\,\beta^7+6\,\alpha^6\,\beta^4+6\,\alpha^4\,\beta^6+6\,\alpha^5\,\beta^5+3\,\alpha^7\,\beta^3-23\,\alpha^3\,\beta^6 \\ &-23\,\alpha^6\,\beta^3-20\,\alpha^6\,\beta-20\,\alpha\,\beta^6+33\,\alpha^2\,\beta^6+33\,\alpha^6\,\beta^2+4\,\alpha^6+4\,\beta^6 \\ &+3\,\alpha\,\beta^7-6\,\alpha^2\,\beta^7-10\,\beta^5-10\,\alpha^5-6\,\alpha^7\,\beta^2+124\,\alpha^3\,\beta^2-12\,\alpha^3\,\beta \\ &+124\,\alpha^2\,\beta^3-36\,\alpha^2\,\beta^2-12\,\alpha\,\beta^3-276\,\alpha^3\,\beta^3 \\ &-110\,\alpha^4\,\beta^4+6\,\alpha^4+6\,\beta^4)\Big/((-\beta-\alpha+\alpha\,\beta)\,(-1+\alpha)^2\,(-1+\beta)^2) \end{aligned} \tag{B.32}$$

$$T_{4,2}(\alpha,\beta) =(3\,\alpha^4\,\beta-\alpha^4+15\,\alpha^3\,\beta^2-20\,\alpha^3\,\beta+7\,\alpha^3-30\,\alpha^2\,\beta^2+23\,\alpha^2\,\beta-10\,\alpha^2+15\,\alpha^2\,\beta^3 + 3\,\alpha\,\beta^4-20\,\alpha\,\beta^3-4\,\alpha\,\beta+23\,\alpha\,\beta^2-\beta^4+7\,\beta^3-10\,\beta^2)/((-1+\beta)\,(-1+\alpha)) \tag{B.33}$$

$$\begin{aligned}T_{4,1}(\alpha,\beta) =&-2(10\,\alpha^4\,\beta^3+35\,\alpha^4\,\beta+10\,\alpha^3\,\beta^4-34\,\alpha^2\,\beta^4+35\,\alpha\,\beta^4-34\,\alpha^4\,\beta^2-3\,\alpha\,\beta^5\\ &-3\,\alpha^5\,\beta+2\,\alpha^5\,\beta^2+2\,\alpha^2\,\beta^5+23\,\alpha^3+\beta^5+\alpha^5-24\,\alpha\,\beta-12\,\beta^2-12\,\alpha^2\\ &+114\,\alpha^3\,\beta^2-86\,\alpha^3\,\beta+114\,\alpha^2\,\beta^3-152\,\alpha^2\,\beta^2+79\,\alpha^2\,\beta-86\,\alpha\,\beta^3+79\,\alpha\,\beta^2\\ &+23\,\beta^3-58\,\alpha^3\,\beta^3-12\,\alpha^4-12\,\beta^4)\Big/((-1+\alpha)^2\,(-1+\beta)^2)\end{aligned} \tag{B.34}$$

$$T_{4,0}(\alpha,\beta) =22\,\beta^2+22\,\alpha^2-5\,\alpha^2\,\beta-5\,\alpha\,\beta^2+28\,\alpha\,\beta-\alpha^3-\beta^3 \tag{B.35}$$

$$\begin{aligned}S_{5,2}(\alpha,\beta) =&-(-\alpha^2+\alpha^2\,\beta-\alpha\,\beta+\alpha\,\beta^2-\beta^2)(6\,\alpha+6\,\beta-6\,\alpha^2-6\,\beta^2-30\,\alpha\,\beta-36\,\alpha^2\,\beta^2\\ &+32\,\alpha^2\,\beta+32\,\alpha\,\beta^2+\beta^3+\alpha^3+9\,\alpha^2\,\beta^3+9\,\alpha^3\,\beta^2-9\,\alpha\,\beta^3-9\,\alpha^3\,\beta)/((-1+\beta)\\ &(-1+\alpha)\,(-\beta-\alpha+\alpha\,\beta))\end{aligned} \tag{B.36}$$

$$\begin{aligned}\widetilde{S}_{5,1}(\alpha,\beta) =&-(-5\,\beta^3+52\,\alpha^2\,\beta^2-35\,\alpha^2\,\beta^3+26\,\alpha\,\beta^3+26\,\alpha^3\,\beta-19\,\alpha\,\beta^2+6\,\alpha^2\,\beta^4\\ &-35\,\alpha^3\,\beta^2-5\,\alpha^3+6\,\alpha^4\,\beta^2-19\,\alpha^2\,\beta+2\,\alpha^4-7\,\alpha^4\,\beta+12\,\alpha^3\,\beta^3+2\,\beta^4\\ &-7\,\alpha\,\beta^4)/(-\beta-\alpha+\alpha\,\beta)\end{aligned} \tag{B.37}$$

$$\begin{aligned}S_{5,1}(\alpha,\beta) =&(-133\,\alpha^4\,\beta^3-89\,\alpha^4\,\beta-133\,\alpha^3\,\beta^4+176\,\alpha^2\,\beta^4-89\,\alpha\,\beta^4+176\,\alpha^4\,\beta^2+18\,\alpha^3\,\beta^5\\ &+18\,\alpha^5\,\beta^3+17\,\alpha\,\beta^5+17\,\alpha^5\,\beta-35\,\alpha^5\,\beta^2-35\,\alpha^2\,\beta^5-12\,\alpha^3-319\,\alpha^3\,\beta^2\\ &+136\,\alpha^3\,\beta-319\,\alpha^2\,\beta^3+236\,\alpha^2\,\beta^2-60\,\alpha^2\,\beta+136\,\alpha\,\beta^3-60\,\alpha\,\beta^2-12\,\beta^3\\ &+310\,\alpha^3\,\beta^3+36\,\alpha^4\,\beta^4+10\,\alpha^4+10\,\beta^4)/(2\,(-1+\beta)\,(-1+\alpha)\,(-\beta-\alpha+\alpha\,\beta))\end{aligned} \tag{B.38}$$

$$\begin{aligned}S_{5,0}(\alpha,\beta) =&-(5\,\alpha^4\,\beta^3+6\,\alpha^4\,\beta+5\,\alpha^3\,\beta^4-20\,\alpha^2\,\beta^4+6\,\alpha\,\beta^4-40\,\alpha^3\,\beta^3+\alpha^5\,\beta^2-20\,\alpha^4\,\beta^2\\ &-2\,\alpha\,\beta^5-2\,\alpha^5\,\beta+\alpha^2\,\beta^5+28\,\alpha^3\,\beta^2+28\,\alpha^2\,\beta^3+2\,\beta^5+2\,\alpha^5)/(2\,\alpha\,\beta\\ &(-\beta-\alpha+\alpha\,\beta))\end{aligned} \tag{B.39}$$

$$\begin{aligned}D_{5,1}(\alpha,\beta) =&-(-178\,\alpha^4\,\beta^3+2\,\alpha^4\,\beta-178\,\alpha^3\,\beta^4+112\,\alpha^2\,\beta^4+2\,\alpha\,\beta^4+112\,\alpha^4\,\beta^2\\ &-28\,\alpha^3\,\beta^5-28\,\alpha^5\,\beta^3-33\,\alpha\,\beta^5-33\,\alpha^5\,\beta+45\,\alpha^5\,\beta^2+10\,\alpha^5\,\beta^4+10\,\alpha^4\,\beta^5\\ &+45\,\alpha^2\,\beta^5+14\,\alpha^3\,\beta^6+14\,\alpha^6\,\beta^3+13\,\alpha^6\,\beta+13\,\alpha\,\beta^6-27\,\alpha^2\,\beta^6-27\,\alpha^6\,\beta^2\\ &+6\,\beta^5+6\,\alpha^5-200\,\alpha^3\,\beta^2+24\,\alpha^3\,\beta-200\,\alpha^2\,\beta^3+64\,\alpha^2\,\beta^2+24\,\alpha\,\beta^3\\ &+370\,\alpha^3\,\beta^3+62\,\alpha^4\,\beta^4-8\,\alpha^4-8\,\beta^4)/(2\,(-1+\beta)\,(-1+\alpha)\,(-\beta-\alpha+\alpha\,\beta))\end{aligned} \tag{B.40}$$

$$
\begin{aligned}
T_{5,2}(\alpha,\beta) =& (18\,\alpha^5\,\beta^3 - 24\,\alpha^5\,\beta^2 + 5\,\alpha^5\,\beta + \alpha^5 + 36\,\alpha^4\,\beta^4 - 84\,\alpha^4\,\beta^3 + 78\,\alpha^4\,\beta^2 - 25\,\alpha^4\,\beta \\
& - \alpha^4 + 18\,\alpha^3\,\beta^5 + 98\,\alpha^3\,\beta^3 - 60\,\alpha^3\,\beta^2 + 24\,\alpha^3\,\beta - 84\,\alpha^3\,\beta^4 - 24\,\alpha^2\,\beta^5 \\
& + 78\,\alpha^2\,\beta^4 + 2\,\alpha^2\,\beta^2 - 60\,\alpha^2\,\beta^3 + 5\,\alpha\,\beta^5 - 25\,\alpha\,\beta^4 + 24\,\alpha\,\beta^3 \\
& + \beta^5 - \beta^4)/(2\,\alpha\,\beta\,(-1+\alpha)(-1+\beta))
\end{aligned}
\tag{B.41}
$$

$$
\begin{aligned}
T_{5,1}(\alpha,\beta) =& -(12\,\alpha^5\,\beta^3 - 20\,\alpha^5\,\beta^2 + 9\,\alpha^5\,\beta - \alpha^5 + 24\,\alpha^4\,\beta^4 - 88\,\alpha^4\,\beta^3 + 94\,\alpha^4\,\beta^2 \\
& - 31\,\alpha^4\,\beta + \alpha^4 + 12\,\alpha^3\,\beta^5 + 178\,\alpha^3\,\beta^3 - 124\,\alpha^3\,\beta^2 + 24\,\alpha^3\,\beta - 88\,\alpha^3\,\beta^4 \\
& - 20\,\alpha^2\,\beta^5 + 94\,\alpha^2\,\beta^4 + 46\,\alpha^2\,\beta^2 - 124\,\alpha^2\,\beta^3 + 9\,\alpha\,\beta^5 - 31\,\alpha\,\beta^4 \\
& + 24\,\alpha\,\beta^3 - \beta^5 + \beta^4)/(\alpha\,\beta(-1+\alpha)\,(-1+\beta))
\end{aligned}
\tag{B.42}
$$

$$
\begin{aligned}
T_{5,0}(\alpha,\beta) =& -(7\,\beta^4 + 6\,\alpha^4\,\beta^2 + 32\,\alpha^3\,\beta + 6\,\alpha^2\,\beta^4 + 32\,\alpha\,\beta^3 + 12\,\alpha^3\,\beta^3 - 14\,\alpha^4\,\beta + 7\,\alpha^4 \\
& + 66\,\alpha^2\,\beta^2 - 70\,\alpha^3\,\beta^2 - 14\,\alpha\,\beta^4 - 70\,\alpha^2\,\beta^3)/(2\,\alpha\,\beta)
\end{aligned}
\tag{B.43}
$$

C. Real–emission coefficients at NLO and NNLO

Below we list the real–emission coefficients $c_i^{(n)}(\alpha, \beta = r\alpha)$ in eq. (4.1) at NLO ($n = 1$) and NNLO ($n = 2$) of each of the five structure functions, $i = 1$ to 5. Note that these expressions include a singular (non-integrable) piece for $r \to 0$. According to the default subtraction prescription we use, eq. (6.4), the plus prescription is defined with respect to $r = \beta/\alpha$. The coefficients $c_i^{(n)}(\alpha, \beta = r\alpha)$ are therefore understood to be separated as in eq. (6.6), where the the singular part that is put under the plus prescription as given in eq. (6.7). The NLO coefficients are:

$$
\begin{aligned}
c_1^{(1)}(\alpha,\alpha r) =& \frac{\left(2\alpha^2 - 10\alpha + 7\right) r^3 + \left(2\alpha^2 - 4\alpha - 7\right) r^2 - (10\alpha + 7)r + 7}{4(r-1)r} \\
& - \frac{\left(r^4 + r^2\right)\alpha^2 - 2r\left(r^3 + 2r^2 + 2r + 1\right)\alpha + 2\left(r^2 - 1\right)^2}{2(r-1)^2 r}\ln r
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
c_2^{(1)}(\alpha,\alpha r) =& \frac{\left(-6\alpha^3 + 20\alpha^2 - 21\alpha + 7\right) r^3 - \left(6\alpha^3 - 56\alpha^2 + 57\alpha - 10\right) r^2}{\alpha(r-1)^3} \\
& + \frac{\left(20\alpha^2 - 57\alpha + 38\right) r}{\alpha(r-1)^3} + \frac{(10 - 21\alpha)r + 7}{\alpha(r-1)^3 r} \\
& + 2\left[\frac{\left(\alpha^3 - 3\alpha^2 + 4\alpha - 2\right) r^4 + \left(4\alpha^3 - 21\alpha^2 + 18\alpha - 2\right) r^3}{\alpha(r-1)^4}\right. \\
& \left. + \frac{\left(\alpha^3 - 21\alpha^2 + 34\alpha - 14\right) r^3 + \left(-3\alpha^2 + 18\alpha - 14\right) r^2 + (4\alpha - 2)r - 2}{\alpha(r-1)^4 r}\right]\ln r
\end{aligned}
\tag{C.2}
$$

$$c_3^{(1)}(\alpha,\alpha r) = \frac{\left(4\alpha^2-10\alpha+7\right)r^2+(2-10\alpha)r+7}{2\alpha(r-1)r} - \frac{\left(\alpha^2-2\alpha+2\right)r^3+\left(\alpha^2-6\alpha+2\right)r^2-2(\alpha-1)r+2}{\alpha(r-1)^2 r}\ln r \tag{C.3}$$

$$c_4^{(1)}(\alpha,\alpha r) = \frac{22r^2+28r-\alpha\left(r^3+5r^2+5r+1\right)+22}{\alpha\,(r-1)^3} + \frac{4\left((\alpha-2)r^3+(\alpha-7)r^2+(\alpha-7)r-2\right)}{\alpha\,(r-1)^4}\ln r \tag{C.4}$$

$$c_5^{(1)}(\alpha,\alpha r) = \frac{\left(-6\alpha^2+14\alpha-7\right)r^3-2\left(6\alpha^2-35\alpha+16\right)r^2-\left(6\alpha^2-70\alpha+66\right)r}{2\alpha(r-1)^3} + \frac{2(7\alpha-16)r-7}{2\alpha(r-1)^3 r} + \left[\frac{\left(\alpha^2-2\alpha+2\right)r^4+\left(5\alpha^2-20\alpha+6\right)r^3}{\alpha(r-1)^4} + \frac{\left(5\alpha^2-40\alpha+28\right)r^2+\left(\alpha^2-20\alpha+28\right)r}{\alpha(r-1)^4} + \frac{-2(\alpha-3)r+2}{\alpha(r-1)^4 r}\right]\ln r \tag{C.5}$$

The NNLO coefficients in the large–β_0 limit are:

$$c_i^{(2)}(\alpha,\alpha r) = \frac{-1}{[\alpha(1-r)]^{2y_i}}\left\{\mathbb{A}_i(\alpha,r)\,\ln(r) + \mathbb{B}_i(\alpha,r)\,\frac{\alpha(1-r)(1+r-\alpha r)}{r}\,\ln(1+r-\alpha r) + \mathbb{T}_i(\alpha,r) + 2\alpha(1+r-\alpha r)\,S_{i,0}(\alpha,\alpha r)\left[\mathrm{Li}_2\left(\frac{1-\alpha r}{1+r-\alpha r}\right) - \mathrm{Li}_2\left(\frac{r\,(1-\alpha)}{1+r-\alpha r}\right) + \frac{1}{2}\ln^2(r) - \ln(r)\,\ln(1+r-\alpha r)\right]\right\} + \left(\frac{5}{3} - 2\ln(\alpha) - \ln(r)\right)c_i^{(1)}(\alpha,\alpha r) \tag{C.6}$$

with

$$\mathbb{T}_i(\alpha,r) = \alpha(1-r)\left[T_{i,1}(\alpha,\alpha r) + \frac{1}{2}T_{i,2}(\alpha,\alpha r)\right]$$

where $T_{i,j}(\alpha,\beta)$ and $S_{i,0}(\alpha,\beta)$ in (C.6) are given in Appendix B and $y_i = [1,2,1,2,2]$. The other functions $\mathbb{A}_i(\alpha,r)$ and $\mathbb{B}_i(\alpha,r)$ are:

$$\mathbb{A}_1(\alpha,r) = \frac{\alpha^2}{2r(\alpha r-1)^2}\left[\left(2\alpha^4-4\alpha^3+3\alpha^2\right)r^6+\left(2\alpha^4-7\alpha^3+6\alpha^2-5\alpha\right)r^5 + \left(\alpha^4-11\alpha^3+8\alpha^2+5\alpha+2\right)r^4+\left(19\alpha^2-8\alpha-6\right)r^3 + \left(-5\alpha^2-12\alpha+4\right)r^2+(6\alpha+2)r-2\right] \tag{C.7}$$

$$\mathbb{A}_2(\alpha,r) = \alpha^3\left[\left(-8\alpha^3+20\alpha^2-18\alpha+6\right)r^4+\left(-20\alpha^3+78\alpha^2-84\alpha+26\right)r^3 - \left(2\alpha^3-36\alpha^2+76\alpha-36\right)r^2+\left(-2\alpha^2-16\alpha+20\right)r+8\alpha-\frac{4}{r}\right] \tag{C.8}$$

$$\mathbb{A}_3(\alpha,r)=\frac{\alpha}{r(\alpha r-1)}\Big[\left(3\alpha^3-4\alpha^2+3\alpha\right)r^4+\left(\alpha^3-8\alpha^2+6\alpha-2\right)r^3 + \left(\alpha^2+6\alpha-4\right)r^2-4\alpha r+2\Big] \tag{C.9}$$

$$\begin{aligned}\mathbb{A}_4(\alpha,r)=\frac{\alpha^3}{(\alpha r-1)^3}\Big[&-\alpha^4r^7-\left(10\alpha^4-21\alpha^3+4\alpha^2\right)r^6-\left(13\alpha^4-64\alpha^3+68\alpha^2-6\alpha\right)r^5\\ &+\left(-6\alpha^4+67\alpha^3-138\alpha^2+72\alpha-2\right)r^4+\left(10\alpha^3-116\alpha^2+136\alpha-26\right)r^3\\ &+\left(8\alpha^2+76\alpha-48\right)r^2+(-20\alpha-16)r+8\Big]\end{aligned} \tag{C.10}$$

$$\begin{aligned}\mathbb{A}_5(\alpha,r)=\frac{\alpha^3}{r\,(\alpha r-1)^2}\Big[&\left(-4\alpha^4+6\alpha^3-3\alpha^2\right)r^7+\left(-14\alpha^4+47\alpha^3-35\alpha^2+8\alpha\right)r^6\\ &+\left(-11\alpha^4+80\alpha^3-132\alpha^2+52\alpha-4\right)r^5+\left(-\alpha^4+29\alpha^3-133\alpha^2+134\alpha-26\right)r^4\\ &+\left(-20\alpha^2+84\alpha-42\right)r^3+\left(5\alpha^2-2\alpha-18\right)r^2+(4-6\alpha)r+2\Big]\end{aligned} \tag{C.11}$$

$$\begin{aligned}\mathbb{B}_1(\alpha,r)=\frac{\alpha}{2(\alpha-1)^2(\alpha r-1)^2}\Big[&\left(-\alpha^5+7\alpha^4-11\alpha^3+5\alpha^2\right)r^4\\ &+\left(-\alpha^5+4\alpha^4-10\alpha^3+17\alpha^2-9\alpha\right)r^3+\left(7\alpha^4-10\alpha^3-6\alpha^2+3\alpha+4\right)r^2\\ &+\left(-11\alpha^3+17\alpha^2+3\alpha-8\right)r+5\alpha^2-9\alpha+4\Big]\end{aligned} \tag{C.12}$$

$$\mathbb{B}_2(\alpha,r)=2\alpha^2\left[\left(3\alpha^2-8\alpha+5\right)r^3+\left(3\alpha^2-26\alpha+13\right)r^2+(13-8\alpha)r+5\right] \tag{C.13}$$

$$\mathbb{B}_3(\alpha,r)=\frac{\left(-2\alpha^3+6\alpha^2-5\alpha\right)r^2+\left(6\alpha^2-8\alpha+4\right)r-5\alpha+4}{(\alpha-1)(\alpha r-1)} \tag{C.14}$$

$$\begin{aligned}\mathbb{B}_4(\alpha,r)=\frac{\alpha^2}{(\alpha-1)^3(\alpha r-1)^3}\Big[&\left(\alpha^6-2\alpha^5+\alpha^4\right)r^6+\left(5\alpha^6-38\alpha^5+68\alpha^4-37\alpha^3+4\alpha^2\right)r^5\\ &+\left(5\alpha^6-52\alpha^5+171\alpha^4-230\alpha^3+104\alpha^2-6\alpha\right)r^4\\ &+\left(\alpha^6-38\alpha^5+171\alpha^4-306\alpha^3+282\alpha^2-100\alpha+2\right)r^3\\ &-\left(2\alpha^5-68\alpha^4+230\alpha^3-282\alpha^2+160\alpha-34\right)r^2\\ &+\left(\alpha^4-37\alpha^3+104\alpha^2-100\alpha+34\right)r+4\alpha^2-6\alpha+2\Big]\end{aligned} \tag{C.15}$$

$$\begin{aligned}\mathbb{B}_5(\alpha,r) = \frac{\alpha^2}{(\alpha-1)^2(\alpha r-1)^2}\Big[&(\alpha-1)^2\alpha^2(3\alpha-5)r^5\\ &+\alpha\left(6\alpha^4-49\alpha^3+87\alpha^2-55\alpha+12\right)r^4\\ &+\left(3\alpha^5-49\alpha^4+160\alpha^3-180\alpha^2+68\alpha-6\right)r^3\\ &+\left(-11\alpha^4+87\alpha^3-180\alpha^2+140\alpha-30\right)r^2\\ &+\left(13\alpha^3-55\alpha^2+68\alpha-30\right)r-5\alpha^2+12\alpha-6\Big]\end{aligned} \tag{C.16}$$

D. Virtual coefficients at NLO and NNLO

The coefficients $v_i^{(1,2)}(\alpha)$ entering (7.21) for the five structure functions, $i = 1$ to 5 are listed below. These coefficients are computed by expanding eq. (7.20). Let us recall that these expressions correspond to the infrared–subtraction procedure detailed in section 7, where $B[V_0](\alpha,u)|_{\text{sing.}}$ is defined according to (6.12), where the plus prescription is defined with respect to $r = \beta/\alpha$, as in (6.4).

Let us also recall that according to eq. (7.20) there is a simple all–order relation between the virtual corrections for the structure functions $i = 3$ and $i = 1$, namely

$$v_3^{(n)}(\alpha) = \frac{2}{\alpha}\, v_1^{(n)}(\alpha)\,. \tag{D.1}$$

The NLO and NNLO coefficients for the other structure functions are:

$$v_1^{(1)}(\alpha) = \frac{\alpha}{2}\Big[-2\,\mathrm{Li}_2(1-\alpha) - \frac{(-3+2\,\alpha)\ln(\alpha)}{-1+\alpha} - \frac{2\,\pi^2}{3} - \frac{5}{2}\Big] \tag{D.2}$$

$$\begin{aligned}v_1^{(2)}(\alpha) = \frac{\alpha}{2}\Big[&\left(\frac{2\alpha-3}{\alpha-1} - 2\ln(1-\alpha)\right)\ln^2(\alpha) + \left(\frac{33-14\alpha}{6(\alpha-1)} + 2\pi^2\right)\ln(\alpha)\\ &+ \frac{(16-19\,\alpha)\,\mathrm{Li}_2(1-\alpha)}{3(\alpha-1)} - 2\,\mathrm{Li}_3(1-\alpha) - 4\,\mathrm{Li}_3(\alpha) + 2\zeta_3 - \frac{79\,\pi^2}{36} - \frac{71}{12}\Big]\end{aligned} \tag{D.3}$$

$$v_2^{(1)}(\alpha) = -4\ln(\alpha) - 4\,\mathrm{Li}_2(1-\alpha) - \frac{4\,\pi^2}{3} - 5 \tag{D.4}$$

$$\begin{aligned}v_2^{(2)}(\alpha) = (4-4\ln(1-\alpha))\ln^2(\alpha) + \left(4\pi^2 - \frac{2\,(7\alpha-4)}{3(\alpha-1)}\right)\ln(\alpha)\\ - \frac{38}{3}\mathrm{Li}_2(1-\alpha) - 4\,\mathrm{Li}_3(1-\alpha) - 8\,\mathrm{Li}_3(\alpha) + 4\zeta_3 - \frac{79\pi^2}{18} - \frac{71}{6}\end{aligned} \tag{D.5}$$

$$v_4^{(1)}(\alpha) = -\frac{2\,(-1+2\,\alpha)\ln(\alpha)}{(-1+\alpha)^2} + \frac{2}{-1+\alpha} \tag{D.6}$$

$$v_4^{(2)}(\alpha) = \frac{2\,(-1+2\,\alpha)\,\mathrm{Li}_2(1-\alpha)}{(-1+\alpha)^2} + \frac{2\,(-1+2\,\alpha)\,\ln(\alpha)^2}{(-1+\alpha)^2} - \frac{1}{3}\,\frac{(-25+38\,\alpha)\,\ln(\alpha)}{(-1+\alpha)^2} + \frac{19}{3\,(-1+\alpha)} \tag{D.7}$$

$$v_5^{(1)}(\alpha) = 2\,\mathrm{Li}_2(1-\alpha) + \frac{(2\,\alpha^2-2\,\alpha+1)\,\ln(\alpha)}{(-1+\alpha)^2} + \frac{2\,\pi^2}{3} + \frac{5\,\alpha-7}{2\,(-1+\alpha)} \tag{D.8}$$

$$v_5^{(2)}(\alpha) = \left(2\ln(1-\alpha) - \frac{2\,(\alpha-1)\alpha+1}{(\alpha-1)^2}\right)\ln^2(\alpha) + \left(\frac{14\alpha^2+16\alpha-17}{6(\alpha-1)^2} - 2\pi^2\right)\ln(\alpha) - 2\zeta_3 + \frac{22+\alpha\,(19\alpha-44)}{3(\alpha-1)^2}\,\mathrm{Li}_2(1-\alpha) + 2\,\mathrm{Li}_3(1-\alpha) + 4\,\mathrm{Li}_3(\alpha) + \frac{109-71\alpha}{12-12\alpha} + \frac{79\pi^2}{36} \tag{D.9}$$

E. Polynomials entering the triple differential width

Below we list the polynomials in $r = \beta/\alpha$, α and x entering the triple differential width summarized in section 8. The polynomials entering the NLO coefficient through eq. (8.10) are:

$$\begin{aligned}\mathbb{Q}_1(\alpha,\,r,\,x) &= \alpha^2\,(1-\alpha)\,(\alpha^2-2\,\alpha+2)\,r^6 + \alpha\,(1-\alpha)\times \\ &(-7\,\alpha^2+2\,x\,\alpha^2+5\,\alpha-4\,x\,\alpha-6+4\,x)\,r^5 - (-10\,x\,\alpha+42\,\alpha^3-2\,x^2+x^2\,\alpha^3 \\ &+29\,x\,\alpha^2-4+4\,\alpha\,x^2+4\,x\,\alpha^4-3\,x^2\,\alpha^2+4\,\alpha^5-20\,\alpha^4+6\,x+3\,\alpha-28\,x\,\alpha^3 \\ &-25\,\alpha^2)r^4 - (-64\,x\,\alpha+4-98\,\alpha^2+4\,x\,\alpha^4-21\,x^2\,\alpha^2+46\,\alpha-40\,x\,\alpha^3 \\ &-4\,x-20\,\alpha^4+95\,x\,\alpha^2+18\,\alpha\,x^2+68\,\alpha^3+4\,x^2\,\alpha^3)r^3 - (75\,x\,\alpha^2-80\,x\,\alpha \\ &+34\,\alpha\,x^2+32\,\alpha^3+52\,x+\alpha^5-21\,x^2\,\alpha^2+x^2\,\alpha^3-22\,x^2+52\,\alpha+2\,x\,\alpha^4-7\,\alpha^4 \\ &-48\,\alpha^2-30-28\,x\,\alpha^3)r^2 + (-16\,\alpha-18\,\alpha\,x^2+2\,x^2+3\,x^2\,\alpha^2-10-39\,x\,\alpha^2 \\ &-20\,\alpha^3+43\,\alpha^2+8\,x+6\,x\,\alpha^3+44\,x\,\alpha+3\,\alpha^4)r-2\,\alpha^3-(6\,x-13)\,\alpha^2 \\ &-(27+4\,x^2-26\,x)\,\alpha+16+10\,x^2-26\,x\end{aligned} \tag{E.1}$$

$$\mathbb{Q}_2(\alpha,\,r,\,x) = 2\,\alpha^2\,r^2 - 10\,\alpha\,r^2 + 7\,r^2 + 2\,\alpha^2\,r - 4\,\alpha\,r - 7\,r - 10\,\alpha \tag{E.2}$$

$$\mathbb{Q}_3(\alpha,\,r,\,x) = \left(6\,\alpha^3-20\,\alpha^2+21\,\alpha-7\right)r^3 + \left(-17-56\,\alpha^2+57\,\alpha+6\,\alpha^3\right)r^2 + \left(-20\,\alpha^2-17+57\,\alpha\right)r + 21\,\alpha - 31 \tag{E.3}$$

The polynomials entering the NNLO coefficient through eq. (8.11) are:

$$\begin{aligned}\mathbb{P}_1(\alpha,\, r,\, x) = {} & \alpha^3\,(1-\alpha)\,(3\,\alpha^2 - 12\,\alpha + 10)\,r^6 \\ & - \alpha^2\,(38 + 3\,\alpha^4 - 20\,x - 38\,x\,\alpha^2 + 48\,x\,\alpha - 22\,\alpha^3 - 83\,\alpha + 63\,\alpha^2 + 10\,x\,\alpha^3)\,r^5 \\ & - \alpha(26\,\alpha\,x^2 - 46 + 3\,\alpha^5 + 6\,x^2\,\alpha^3 + 4\,x\,\alpha^4 - 10\,x^2 - 58\,\alpha^4 - 140\,x\,\alpha - 82\,x\,\alpha^3 \\ & - 160\,\alpha^2 - 22\,x^2\,\alpha^2 + 170\,x\,\alpha^2 + 155\,\alpha^3 + 110\,\alpha + 48\,x)r^4 - (-334\,\alpha^3 - 28\,x \\ & + 155\,\alpha^4 + 10\,x^2 - 22\,\alpha^5 + 136\,x\,\alpha - 68\,\alpha - 64\,x^2\,\alpha^3 + 242\,\alpha^2 - 52\,\alpha\,x^2 \\ & + 284\,x\,\alpha^3 - 82\,x\,\alpha^4 - 320\,x\,\alpha^2 + 6\,x^2\,\alpha^4 + 100\,x^2\,\alpha^2 + 3\,\alpha^6 + 10\,x\,\alpha^5 + 18)r^3 \\ & + (15\,\alpha^5 + 22\,x^2\,\alpha^3 - 63\,\alpha^4 + 320\,x\,\alpha^2 - 26\,x^2 + 44\,x + 104\,\alpha\,x^2 + 38\,x\,\alpha^4 \\ & - 100\,x^2\,\alpha^2 + 160\,\alpha^3 - 232\,x\,\alpha + 144\,\alpha - 242\,\alpha^2 - 18 - 170\,x\,\alpha^3)r^2 - (136\,x\,\alpha \\ & - 140\,x\,\alpha^2 + 26\,x^2 + 26\,x^2\,\alpha^2 - 44\,x + 18 + 22\,\alpha^4 + 110\,\alpha^2 - 52\,\alpha\,x^2 - 68\,\alpha \\ & - 83\,\alpha^3 + 48\,x\,\alpha^3)r + 10\,\alpha^3 + 2\,(-19 + 10\,x)\,\alpha^2 + 2\,(23 - 24\,x + 5\,x^2)\,\alpha - 18 \\ & + 28\,x - 10\,x^2\end{aligned} \tag{E.4}$$

$$\begin{aligned}\mathbb{P}_2(\alpha,\, r,\, x) = {} & -2\,\alpha^3\,(1-\alpha)\,(34\,\alpha^2 - 74\,\alpha + 59)\,r^7 \\ & - \alpha^2\,(1-\alpha)\,(24\,\alpha^3 - 424\,\alpha^2 + 184\,x\,\alpha^2 + 557\,\alpha - 344\,x\,\alpha - 460 + 236\,x)\,r^6 \\ & + 2\alpha(289\,x + 650\,\alpha - 283 - 1204\,\alpha^2 + 1229\,x\,\alpha^2 + 1291\,\alpha^3 - 822\,x\,\alpha + 52\,x^2\,\alpha^3 \\ & - 804\,x\,\alpha^3 - 539\,\alpha^4 + 157\,\alpha\,x^2 + 76\,x\,\alpha^4 - 59\,x^2 - 150\,x^2\,\alpha^2 + 85\,\alpha^5)r^5 \\ & + (3320\,\alpha^4 - 1208\,x^2\,\alpha^3 + 224 - 667\,\alpha + 200\,x^2\,\alpha^4 - 5680\,\alpha^3 - 342\,x - 4674\,x\,\alpha^2 \\ & + 5352\,x\,\alpha^3 + 118\,x^2 - 554\,\alpha\,x^2 + 3451\,\alpha^2 + 1416\,x\,\alpha - 1732\,x\,\alpha^4 + 24\,\alpha^6 \\ & + 260\,x\,\alpha^5 + 1380\,x^2\,\alpha^2 - 672\,\alpha^5)r^4 + 2(2592\,\alpha^2 + 56 + 2\,x\,\alpha^5 - 127\,\alpha^5 \\ & + 1936\,x\,\alpha^3 + 727\,\alpha^4 - 470\,x\,\alpha^4 + 1120\,x^2\,\alpha^2 - 3424\,x\,\alpha^2 - 2\,x^2\,\alpha^4 - 2105\,\alpha^3 \\ & - 176\,x + 7\,\alpha^6 + 120\,x^2 - 1150\,\alpha + 1910\,x\,\alpha - 820\,\alpha\,x^2 - 310\,x^2\,\alpha^3)r^3 + (16\,\alpha^5 \\ & - 1076\,x + 76\,x^2\,\alpha^3 - 1524\,\alpha\,x^2 + 560\,x^2 + 660\,x^2\,\alpha^2 - 1491\,\alpha^3 + 104\,x\,\alpha^4 \\ & - 2178\,\alpha + 944\,x\,\alpha^3 + 2834\,\alpha^2 - 3424\,x\,\alpha^2 + 303\,\alpha^4 + 516 + 3744\,x\,\alpha)r^2 \\ & - 2(412\,x + 37\,\alpha^4 + 79\,\alpha\,x^2 + 83\,x^2\,\alpha^2 - 218 - 194\,x^2 - 16\,x\,\alpha^2 + 123\,x\,\alpha^3 - 493\,x\,\alpha \\ & + 401\,\alpha - 184\,\alpha^2 - 36\,\alpha^3)r + 44\,\alpha^3 + (-145 + 138\,x)\,\alpha^2 \\ & + (-296\,x + 94\,x^2 + 165)\,\alpha - 64 - 82\,x^2 + 146\,x\end{aligned} \tag{E.5}$$

$$\begin{aligned}\mathbb{P}_3(\alpha,\, r,\, x) = {} & \alpha\,(1-\alpha)\,(50\,\alpha^2 - 142\,\alpha + 85)\,r^3 - (50\,\alpha^4 + 219\,\alpha^2 - 142\,\alpha + 85 - 200\,\alpha^3)\,r^2 \\ & + (-304\,\alpha^2 + 192\,\alpha^3 + 39\,\alpha + 85)\,r - 142\,(-1+\alpha)\,\alpha\end{aligned} \tag{E.6}$$

$$\begin{aligned}\mathbb{P}_4(\alpha,\, r,\, x) = {} & (1-\alpha)\,(150\,\alpha^2 - 218\,\alpha + 85)\,r^3 - (-1016\,\alpha^2 + 1035\,\alpha - 287 + 150\,\alpha^3)\,r^2 \\ & + (368\,\alpha^2 - 1035\,\alpha + 395)\,r + 457 - 303\,\alpha\end{aligned} \tag{E.7}$$

$$\mathbb{P}_5(\alpha,\, r,\, x) = -(85 + 88\,\alpha^2 - 142\,\alpha)\,r + 142\,\alpha - 123 \tag{E.8}$$

References

[1] Heavy Flavor Averaging Group (HFAG) collaboration, E. Barberio et al., *Averages of B-hadron properties at the end of 2005*, hep-ex/0603003.

[2] Heavy Flavor Averaging Group (HFAG) collaboration, *Semileptonic decays*, http://www.slac.stanford.edu/xorg/hfag/.

[3] CKMfitter Group collaboration, J. Charles et al., *CP-violation and the ckm matrix: assessing the impact of the asymmetric B factories*, *Eur. Phys. J.* **C 41** (2005) 1 [hep-ph/0406184].

[4] UTfit collaboration, M. Bona et al., *The 2004 UTfit collaboration report on the status of the unitarity triangle in the standard model*, *JHEP* **07** (2005) 028 [hep-ph/0501199].

[5] M. Neubert, *Analysis of the photon spectrum in inclusive $B \to X_s\gamma$ decays*, *Phys. Rev.* **D 49** (1994) 4623 [hep-ph/9312311].

[6] M. Neubert, *QCD based interpretation of the lepton spectrum in inclusive $\bar{B} \to X_u$ lepton anti-neutrino decays*, *Phys. Rev.* **D 49** (1994) 3392 [hep-ph/9311325].

[7] I.I.Y. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, *On the motion of heavy quarks inside hadrons: universal distributions and inclusive decays*, *Int. J. Mod. Phys.* **A 9** (1994) 2467 [hep-ph/9312359].

[8] G.P. Korchemsky and G. Sterman, *Infrared factorization in inclusive B meson decays*, *Phys. Lett.* **B 340** (1994) 96 [hep-ph/9407344].

[9] A.F. Falk, E. Jenkins, A.V. Manohar and M.B. Wise, *QCD corrections and the endpoint of the lepton spectrum in semileptonic B decays*, *Phys. Rev.* **D 49** (1994) 4553 [hep-ph/9312306].

[10] C.W. Bauer and A.V. Manohar, *Shape function effects in $B \to X_s\gamma$ and $B \to X_u l\nu$ decays*, *Phys. Rev.* **D 70** (2004) 034024 [hep-ph/0312109].

[11] B.O. Lange, M. Neubert and G. Paz, *Theory of charmless inclusive B decays and the extraction of v(ub)*, *Phys. Rev.* **D 72** (2005) 073006 [hep-ph/0504071].

[12] B.O. Lange, M. Neubert and G. Paz, *A two-loop relation between inclusive radiative and semileptonic b decay spectra*, *JHEP* **10** (2005) 084 [hep-ph/0508178].

[13] U. Aglietti, G. Ferrera and G. Ricciardi, *Semi-inclusive b decays and a model for soft-gluon effects*, hep-ph/0608047.

[14] U. Aglietti, G. Ricciardi and G. Ferrera, *Threshold resummed spectra in $B \to X_u l\nu$ decays in NLO, I*, *Phys. Rev.* **D 74** (2006) 034004 [hep-ph/0507285].

[15] U. Aglietti, *Resummed $B \to X_u l\nu$ decay distributions to next-to- leading order*, *Nucl. Phys.* **B 610** (2001) 293 [hep-ph/0104020].

[16] J.R. Andersen and E. Gardi, *Taming the $B \to X_s\gamma$ spectrum by dressed gluon exponentiation*, *JHEP* **06** (2005) 030 [hep-ph/0502159].

[17] J.R. Andersen and E. Gardi, *Inclusive spectra in charmless semileptonic b decays by dressed gluon exponentiation*, *JHEP* **01** (2006) 097 [hep-ph/0509360].

JHEP12(2006)036

[18] J.R. Andersen and E. Gardi, *Radiative b decay spectrum: DGE at NNLO*, hep-ph/0609250.

[19] E. Gardi and J.R. Andersen, *Progress in computing inclusive B decay spectra*, *Nucl. Phys.* **157** *(Proc. Suppl.)* (2006) 207 [hep-ph/0601181].

[20] E. Gardi, *Inclusive distributions near kinematic thresholds*, *ECONF* **C0601121** (2006) E003 [hep-ph/0606080].

[21] S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, *On the elimination of scale ambiguities in perturbative quantum chromodynamics*, *Phys. Rev.* **D 28** (1983) 228.

[22] T. van Ritbergen, *The second order QCD contribution to the semileptonic $B \to u$ decay rate*, *Phys. Lett.* **B 454** (1999) 353 [hep-ph/9903226].

[23] M.E. Luke, M.J. Savage and M.B. Wise, *Perturbative strong interaction corrections to the heavy quark semileptonic decay rate*, *Phys. Lett.* **B 343** (1995) 329 [hep-ph/9409287].

[24] I.I.Y. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, *The pole mass of the heavy quark. Perturbation theory and beyond*, *Phys. Rev.* **D 50** (1994) 2234 [hep-ph/9402360].

[25] M. Beneke and V.M. Braun, *Heavy quark effective theory beyond perturbation theory: renormalons, the pole mass and the residual mass term*, *Nucl. Phys.* **B 426** (1994) 301 [hep-ph/9402364].

[26] M. Beneke, V.M. Braun and V.I. Zakharov, *Bloch-nordsieck cancellations beyond logarithms in heavy particle decays*, *Phys. Rev. Lett.* **73** (1994) 3058 [hep-ph/9405304].

[27] E. Gardi, *Radiative and semi-leptonic b-meson decay spectra: Sudakov resummation beyond logarithmic accuracy and the pole mass*, *JHEP* **04** (2004) 049 [hep-ph/0403249].

[28] S.W. Bosch, B.O. Lange, M. Neubert and G. Paz, *Factorization and shape-function effects in inclusive B-meson decays*, *Nucl. Phys.* **B 699** (2004) 335 [hep-ph/0402094].

[29] P. Gambino, G. Ossola and N. Uraltsev, *Hadronic mass and Q^2 moments of charmless semileptonic B decay distributions*, *JHEP* **09** (2005) 010 [hep-ph/0505091].

[30] K. Melnikov and A. Mitov, *The photon energy spectrum in $b \to x/s$ + gamma in perturbative QCD through $O(\alpha_s^2)$*, *Phys. Lett.* **B 620** (2005) 69 [hep-ph/0505097].

[31] H.M. Asatrian, T. Ewerth, A. Ferroglia, P. Gambino and C. Greub, *Magnetic dipole operator contributions to the photon energy spectrum in $\bar{B} \to X_s\gamma$ at $O(\alpha_s^2)$*, hep-ph/0607316.

[32] Z. Ligeti, M.E. Luke, A.V. Manohar and M.B. Wise, *The $\bar{B} \to X_s\gamma$ photon spectrum*, *Phys. Rev.* **D 60** (1999) 034019 [hep-ph/9903305].

[33] F. De Fazio and M. Neubert, *$B \to X_u l\bar{\nu}_l$ decay distributions to order α_s*, *JHEP* **06** (1999) 017 [hep-ph/9905351].

[34] A.H. Hoang, Z. Ligeti and M. Luke, *Perturbative corrections to the determination of v(ub) from the p+ spectrum in $B \to X_u l\nu$*, *Phys. Rev.* **D 71** (2005) 093007 [hep-ph/0502134].

[35] V. Aquila, P. Gambino, G. Ridolfi and N. Uraltsev, *Perturbative corrections to semileptonic b decay distributions*, *Nucl. Phys.* **B 719** (2005) 77 [hep-ph/0503083].

[36] G.P. Korchemsky and G. Marchesini, *Structure function for large x and renormalization of Wilson loop*, *Nucl. Phys.* **B 406** (1993) 225 [hep-ph/9210281].

[37] E. Gardi, *On the quark distribution in an on-shell heavy quark and its all-order relations with the perturbative fragmentation function*, *JHEP* **02** (2005) 053 [hep-ph/0501257].

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[38] S. Moch, J.A.M. Vermaseren and A. Vogt, *The three-loop splitting functions in QCD: the non-singlet case*, *Nucl. Phys.* **B 688** (2004) 101 [hep-ph/0403192].

[39] S.J. Brodsky, E. Gardi, G. Grunberg and J. Rathsman, *Disentangling running coupling and conformal effects in QCD*, *Phys. Rev.* **D 63** (2001) 094017 [hep-ph/0002065].

[40] Y.L. Dokshitzer, G. Marchesini and B.R. Webber, *Dispersive approach to power-behaved contributions in QCD hard processes*, *Nucl. Phys.* **B 469** (1996) 93 [hep-ph/9512336].

[41] M. Beneke and V.M. Braun, *Renormalons and power corrections*, hep-ph/0010208.

[42] M. Beneke, *Renormalons*, *Phys. Rept.* **317** (1999) 1 [hep-ph/9807443].

[43] P. Ball, M. Beneke and V.M. Braun, *Resummation of $(\beta^0\alpha_s)^n$ corrections in QCD: techniques and applications to the tau hadronic width and the heavy quark pole mass*, *Nucl. Phys.* **B 452** (1995) 563 [hep-ph/9502300].

[44] P. Ball, M. Beneke and V.M. Braun, *Resummation of running coupling effects in semileptonic b meson decays and extraction of $|v(cb)|$*, *Phys. Rev.* **D 52** (1995) 3929 [hep-ph/9503492].

[45] E. Gardi and G. Grunberg, *Power corrections in the single dressed gluon approximation: the average thrust as a case study*, *JHEP* **11** (1999) 016 [hep-ph/9908458].

[46] E. Gardi, J. Kuokkanen, K. Rummukainen and H. Weigert, *Running coupling and power corrections in nonlinear evolution at the high-energy limit*, hep-ph/0609087.

[47] M.Y. Kalmykov, *Gauss hypergeometric function: reduction, epsilon-expansion for integer/half-integer parameters and Feynman diagrams*, *JHEP* **04** (2006) 056 [hep-th/0602028].

[48] G. Grunberg, *The renormalization scheme invariant borel transform and the qed renormalons*, *Phys. Lett.* **B 304** (1993) 183.

[49] E. Gardi and R.G. Roberts, *The interplay between sudakov resummation, renormalons and higher twist in deep inelastic scattering*, *Nucl. Phys.* **B 653** (2003) 227 [hep-ph/0210429].

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